hand, all the root of $\lambda^n - 1$ forms a subgroup of $F_p^* = F_p \backslash \{0\}$. Thus n divides $p - 1$.

Conversely, if n divides $p-1$, $\lambda^n - 1$ has no multiple root and

$$\lambda^n - 1 = (\lambda - 1) \cdot (\lambda - a^d)(\lambda - a^{2d}) \cdots (\lambda - a^{(n-1)d})$$

where $d = \frac{p-1}{n}$ and a is the generator of the group F_p^*. Hence M is similar to $\text{diag}\{1, a^d, \cdots, a^{(n-1)d}\}$ in $M_n(F_p)$. So Π is diagonalizable as a linear map over F_p.

1109

Let $A(t)$ be a non-singular matrix whose elements are differentiable functions of real variable t. Let $A'(t)$ denote the matrix formed by the derivatives of the elements. Show that the derivative of the determinant $\det A$ satisfies

$$\frac{d}{dt}(\det A) = \det A \cdot \text{trace}(A' \cdot A^{-1}).$$

(Harvard)

Solution.

Let

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

Then

$$A'(t) = \begin{pmatrix} a'_{11}(t) & a'_{12}(t) & \cdots & a'_{1n}(t) \\ a'_{21}(t) & a'_{22}(t) & \cdots & a'_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a'_{n1}(t) & a'_{n2}(t) & \cdots & a'_{nn}(t) \end{pmatrix}$$

and

$$A^{-1} = (\det A)^{-1} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is the algebraic cofactor of $a_{ij}(t)$. Hence

$$\begin{aligned} \det A \cdot \text{trace}(A'A^{-1}) &= \sum_{j=1}^{n} a'_{1j}(t)A_{1j} + \sum_{j=1}^{n} a'_{2j}(t)A_{2j} + \cdots + \sum_{j=1}^{n} a'_{nj}(t)A_{nj} \\ &= \sum_i \sum_j a'_{ij}(t)A_{ij} = \sum_j \left(\sum_i a'_{ij}(t)A_{ij} \right). \end{aligned}$$

For $1 \leq k \leq n$, let

$$A_k = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a'_{k1}(t) & a'_{k2}(t) & \cdots & a'_{kn}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

So,

$$\det A \cdot \text{trace}(A'A^{-1}) = \det A_1 + \det A_2 + \cdots + \det A_n.$$

On the other hand, by definition,

$$\begin{aligned} \frac{d}{dt}(\det A) &= \frac{d}{dt}\left(\sum_{(p_1\cdots p_n)} (-1)^{\tau(p_1\cdots p_n)} a_{1p_1}(t)a_{2p_2}(t)\cdots a_{np_n}(t)\right) \\ &= \sum_{(p_1\cdots p_n)} (-1)^{\tau(p_1\cdots p_n)} \left(\sum_{k=1}^{n} a_{1p_1}(t)\cdots a'_{kp_k}(t)\cdots a_{np_n}(t)\right) \\ &= \sum_{k=1}^{n}\left(\sum_{(p_1\cdots p_n)} (-1)^{\tau(p_1\cdots p_n)} a_{1p_1}(t)\cdots a'_{kp_k}(t)\cdots a_{np_n}(t)\right) \\ &= \det A_1 + \det A_2 + \cdots + \det A_n. \end{aligned}$$

Hence we have proved that

$$\frac{d}{dt}(\det A) = \det A \cdot \text{trace}(A' \cdot A^{-1}).$$

1110

Let V be the vector space of polynomials $p(x) = a + bx + cx^2$ with real coefficients a, b, and c. Define an inner product on V by

$$(p, q) = \frac{1}{2}\int_{-1}^{1} p(x)q(x)dx.$$

(a) Find an orthonormal basis for V consisting of polynomials $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$, having degree $0, 1$, and 2, respectively.

(b) Use the answer to (a) to find the second degree polynomial that solves the minimization problem

$$\min_{p\in V} \int_{-1}^{1} (p(x) - x^3)^2 dx.$$

(Courant Inst.)