For $1 \leq k \leq n$, let

$$A_k = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a'_{k1}(t) & a'_{k2}(t) & \cdots & a'_{kn}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

So,

$$\det A \cdot \operatorname{trace}(A'A^{-1}) = \det A_1 + \det A_2 + \cdots + \det A_n.$$

On the other hand, by definition,

$$\begin{aligned} \frac{d}{dt}(\det A) &= \frac{d}{dt}\left(\sum_{(p_1\cdots p_n)} (-1)^{\tau(p_1\cdots p_n)} a_{1p_1}(t)a_{2p_2}(t)\cdots a_{np_n}(t)\right) \\ &= \sum_{(p_1\cdots p_n)} (-1)^{\tau(p_1\cdots p_n)} \left(\sum_{k=1}^{n} a_{1p_1}(t)\cdots a'_{kp_k}(t)\cdots a_{np_n}(t)\right) \\ &= \sum_{k=1}^{n}\left(\sum_{(p_1\cdots p_n)} (-1)^{\tau(p_1\cdots p_n)} a_{1p_1}(t)\cdots a'_{kp_k}(t)\cdots a_{np_n}(t)\right) \\ &= \det A_1 + \det A_2 + \cdots + \det A_n. \end{aligned}$$

Hence we have proved that

$$\frac{d}{dt}(\det A) = \det A \cdot \operatorname{trace}(A' \cdot A^{-1}).$$

1110

Let V be the vector space of polynomials $p(x) = a + bx + cx^2$ with real coefficients a, b, and c. Define an inner product on V by

$$(p, q) = \frac{1}{2}\int_{-1}^{1} p(x)q(x)dx.$$

(a) Find an orthonormal basis for V consisting of polynomials $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$, having degree $0, 1$, and 2, respectively.

(b) Use the answer to (a) to find the second degree polynomial that solves the minimization problem

$$\min_{p\in V} \int_{-1}^{1} (p(x) - x^3)^2 dx.$$

(*Courant Inst.*)

Solution.

(a) Since $1, x, x^2$ is a base of V, we can orthonormalization $1, x, x^2$ to get an orthonormal basis

$$\phi_0(x) = 1, \quad \phi_1(x) = \sqrt{3}x, \quad \phi_2(x) = -\frac{\sqrt{5}}{2} + \frac{3\sqrt{5}}{2}x^2$$

of V with degree 0, 1 and 2 respectively as usual.

(b) Let

$$p_0(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) \in V \quad (c_i \in I\!R)$$

and

$$q_0(x) = x^3 - p_0(x)$$

such that $q_0(x)$ is orthogonal to V, that is

$$(x^3 - p_0(x), \phi_i(x)) = 0 \quad (i = 0, 1, 2).$$

Then for any $p(x) \in V$,

$$\begin{aligned} \int_{-1}^{1} (p(x) - x^3)^2 dx &= 2|x^3 - p(x)|^2 \\ &= 2|x^3 - p_0(x) + p_0(x) - p(x)|^2 \\ &= 2|q_0(x) + p_0(x) - p(x)|^2. \end{aligned}$$

Since $(q_0(x), p_0(x) - p(x)) = 0$,

$$\int_{-1}^{1} (p(x) - x^3)^2 dx = 2|q_0(x)|^2 + 2|p_0(x) - p(x)|^2.$$

It follows that

$$\min_{p\in V} \int_{-1}^{1} (p(x) - x^3)^2 dx = 2|q_0(x)|^2.$$

Since $(q_0(x), \phi_i(x)) = 0$ if and only if

$$(x^3, \phi_i(x)) = (p_0(x), \phi_i(x)) = c_i \quad (i = 0, 1, 2).$$

By an easy calculation, we get $c_0 = 0$, $c_1 = \frac{\sqrt{3}}{5}$ and $c_2 = 0$. Hence $p_0(x) = \frac{3}{5}x$ and $q_0(x) = x^3 - \frac{3}{5}x$. Obviously,

$$\int_{-1}^{1} (x^3 - \frac{3}{5}x)^2 dx = \frac{8}{175}.$$

Thus, when $p(x) = \frac{3}{5}x$,

$$\int_{-1}^{1} (p(x) - x^3)^2 dx$$

is minimal, and

$$\min_{p \in V} \int_{-1}^{1} (p(x) - x^3)^2 dx = \frac{8}{175}.$$

1111

Suppose that A is an $n \times n$ matrix and that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad y^* = (y_1, \cdots, y_n).$$

Suppose that all the entries of A, x, and y are real.

(a) Show that there exist numbers a and b so that

$$\det(A + sxy^*) = a + bs.$$

(b) Show that if $\det(A) \neq 0$ then $a = \det(A)$ and $b = \det(A) \cdot y^* A^{-1} x$.

(c) Is it true that $a = 0$ if $\det(A) = 0$?

(*Courant Inst.*)

Solution.

(a) Directly,

$$\begin{aligned} & \det(A + sxy^*) \\ = & \det \begin{pmatrix} a_{11} + sx_1y_1 & \cdots & a_{1n} + sx_1y_n \\ \cdots & \cdots & \cdots \\ a_{n1} + sx_ny_1 & \cdots & a_{nn} + sx_ny_n \end{pmatrix} \\ = & \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \\ & + s \cdot \left(\det \begin{pmatrix} x_1y_1 & \cdots & x_1y_n \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ x_2y_1 & \cdots & x_2y_n \\ a_{31} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right. \end{aligned}$$