Chapter 1

An Introduction to Topology

1.1 Preliminary Remarks

1.1.1 Remarks on differential geometry

A physics student is likely to first encounter the subject of differential geometry in a course on general relativity, where spacetime is represented mathematically by a four-dimensional differentiable manifold. However, this is far from being the only use of differential geometry in physics. For example, the Hamiltonian and Lagrangian approaches to classical mechanics are best described in this way; and the use of differential geometry in quantum field theory has increased steadily in recent decades—for example, in canonical quantum gravity, superstring theory, the non-linear σ-model, topological quantum field theory, and Yang-Mills theory.

Evidently, no excuse is needed for teaching a course on differential geometry to postgraduate students of theoretical physics. However, the impression of the subject gained from, say, an undergraduate course in general relativity can be rather misleading when viewed from the perspective of modern mathematics. Such courses usually employ a very coordinate-based approach to the subject, with little reference to the fact that more than one coordinate system may be needed to cover a manifold. In particular, although there are usually copious discussions of the effects on tensorial objects of changing from one coordinate system to another, only rarely is it emphasised that the domains of two coordinate systems may differ, and that—for example—the familiar expression involving Jacobian transformations is really only valid on the intersection of the domains of the coordinate systems concerned.

What is neglected in such approaches to differential geometry is the fact that the *topology* of a manifold may be different from that of a vector space, and hence—in particular—it cannot be covered by a single coordinate system. The modern approach to differential geometry is very different: although coordinate systems have an important role to play, the key concepts are developed in a way that is manifestly independent of any specific reference to coordinates. Concomitantly, the fact that a manifold is actually a special type of topological space becomes of greater importance, and for this reason it is appropriate to begin any text dealing with modern 'coordinate-free' differential geometry with an introduction to general topology and associated ideas. In fact, the subject of topology proper is of considerable significance in many areas of modern theoretical physics, and is well worth studying in its own right.

1.1.2 Remarks on topology

The subject of topology can be approached in a variety of ways. At the most abstract level, a 'topology' on a set X consists of a collection of subsets of X—known as the *open sets* of the topology—that satisfy certain axioms (they are listed in Theorem 1.3). This special collection of subsets is then used to give a purely set-theoretic notion of characteristic topological ideas such as 'nearness', 'convergence of a sequence', 'continuity of a function' *etc.* From a physical perspective, one could say that topology is concerned with the relation between points and 'regions': in particular, open sets are what 'real things' can exist in.

Many excellent books on topology take an abstract approach from the outset[1]. However, on a first encounter with the idea of a topology, it is not obvious why that particular set of axioms is chosen rather

---

[1] Two classic examples are Bourbaki (1966) and Kelly (1970).

than any other, and the underlying motivation only slowly becomes clear. For this reason, the particular introduction to general topology given in Section 1.4 is aimed at motivating the axioms for topology by starting with the broadest structure one can conceive with respect to which the notion of a converging sequence makes sense, and then to show how this definition is narrowed to give the standard axioms for general topology.

Other texts take a somewhat different approach and motivate the axioms for topology by starting first with a *metric* space: a special type of topological space whose underlying ideas are more intuitively accessible than are those of topology in general. In addition, metric spaces play many important roles in theoretical physics in their own right; and for these reasons we shall begin with a short introduction to the theory of such spaces. But it should be emphasised that, in general, what follows cannot be regarded as a comprehensive introduction to topology, and it should be supplemented with private study. The most I can do in the limited space available is to provide a quick introduction to some of the key ideas. However, I have also included topics that I feel are of potential interest in theoretical physics but which do not appear in the standard texts on topology: a good example is the lattice structure on the set of all topologies on a given set.

1.2 Metric Spaces

1.2.1 The simple idea of convergence

A key ingredient in any topological-type structure on a set X is the sense in which a point[2] $x \in X$ can be said to be 'near' to another point $y \in X$—without such a concept, the points in X are totally disconnected from each other. In particular, we would like to say that an infinite sequence $(x_1, x_2, \ldots)$ of points in X 'converges' to a point $x \in X$ if the elements of the sequence get arbitrarily near to x in an appropriate way. We shall use the idea of the convergence of

---

[2] The notation $x \in X$ means that x is an element of the set X.