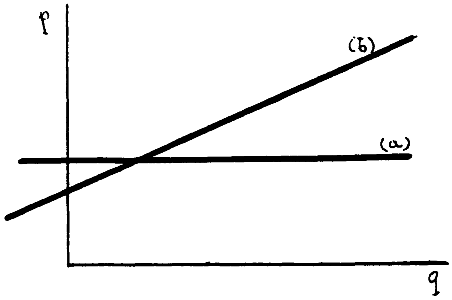

Figure 1.1: Arbitrariness of p

The above process is not unique. For example, for a free particle ($\ddot{q} = 0$) rather than

$$\dot{p} = 0 \quad , \quad \dot{q} = p \quad , \tag{1.5}$$

according to which p is constant and $q = pt + q_0$ (figure 1.1(a)), we might equivalently have written (figure 1.1(b))

$$\dot{p} = 1 \quad , \quad \dot{q} = p - t \quad , \quad p = t + \alpha \quad , \quad q = \alpha t + \beta \quad . \tag{1.6}$$

As we shall see in Chapter 6, this arbitrariness is connected with one for the Lagrangian.

Once the second order equations have been replaced by systems of first order ones, it is natural to describe the evolution of a system as the motion of a point in a $2n$-dimensional "phase space" for the coordinates $q_1, \dots q_n$ and the momenta $p_1, \dots p_n$.

1.2 Motion of a particle in one dimension

It is instructive to study the motion of a particle in one dimension subject to the elastic force $-kq$ and the repulsive force $+kq$ ($k > 0$), respectively.

In the former case, the equations $\dot{p} = -kq$, $\dot{q} = p/m$ have the general solution ($\omega = \sqrt{k/m}$)

$$q = a\cos(\omega t + \alpha) \quad , \quad p = -m\omega a \sin(\omega t + \alpha) \quad , \tag{1.7}$$

so that the representative point in the phase plane (p, q) moves on an ellipse of semi-axes $a = \sqrt{2E/k}$ and $m\omega a = \sqrt{2mE}$ (figure 1.2), where E is the energy. The point $q = 0$, $p = 0$ describes a state of stable equilibrium ($a = 0$, $E = 0$).

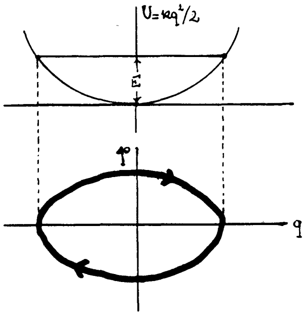

Figure 1.2: Motion under the force $-kq$

In the case of the repulsive force $+kq$, the trajectories in the phase plane are hyperbolae. The solutions

$$q = \pm a \cosh(\omega t) \quad , \quad p = \pm m\omega a \sinh(\omega t) \quad , \tag{1.8}$$

correspond to a particle with initial velocity towards the center of repulsion coming from $q = \pm\infty$ respectively, and stopping at $\pm a$ before rebounding (figure 1.3). In these two cases the energy $E = (p^2/2m) - (kq^2/2)$ is negative $(E = -ka^2/2)$.

The solutions

$$q = \pm a \sinh(\omega t) \quad , \quad p = \pm m\omega a \cosh(\omega t) \quad , \tag{1.9}$$

correspond to the positive energy $E = +ka^2/2$. For $t = 0$ one has $q = 0$, $p = \pm m\omega a$, i.e. a particle right on the center of repulsion with a finite velocity (figure 1.3).

The asymptotes $p = \pm m\omega q$ separate the regions of positive and negative energies. They intersect at the point of unstable equilibrium $p = 0$, $q = 0$.

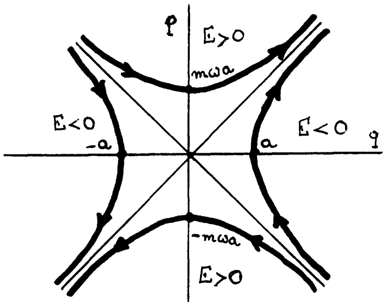

Figure 1.3: Motion under the force $+kq$

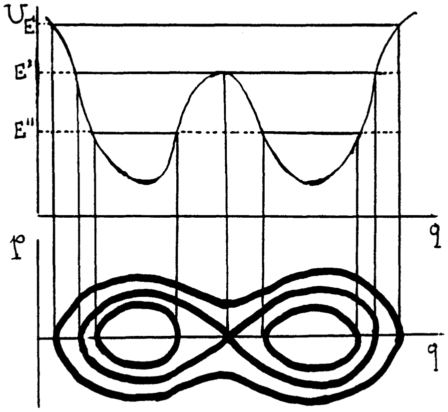

Figure 1.4: Motion near potential energy extrema

Consider now the one-dimensional motion of a particle subject to a conservative force $f(q) = -\mathrm{d}U(q)/\mathrm{d}q$. In the vicinity of a potential energy minimum $q = q_0$, one has $f(q) = -k(q - q_0)$ with $k = (\mathrm{d}^2U/\mathrm{d}q^2)_{q=q_0} > 0$.

If the energy E differs little from $U(q_0)$, the particle moves on an ellipse of center $q = q_0$, $p = 0$ in the (p, q) plane with the period $T = 2\pi\sqrt{m/k}$.

In the vicinity of a potential energy maximum the motion resembles that of a particle subject to the repulsive force $f(q) = +k(q - q_0)$ with $k = -(\mathrm{d}^2U/\mathrm{d}q^2)_{q=q_0} > 0$. Passing through the point $(q = q_0\ ,\ p = 0)$ there will be a critical curve with tangents $p = \pm\sqrt{mk}\,(q - q_0)$, corresponding to the asymptotes of figure 1.3. Such a curve separates regions with $E > U(q_0)$ and $E < U(q_0)$.

Returning to the elastic force $f(q) = -kq$, we notice that the area enclosed by an elliptical trajectory in the (p, q) plane is

$$A = \pi a(m\omega a) = 2\pi E\sqrt{m/k} \quad ,$$

while the period is $T = 2\pi\sqrt{m/k}$. Clearly $T = \mathrm{d}A/\mathrm{d}E$.

This relation is easily generalized to any closed trajectory in the phase plane: $T = \oint \mathrm{d}t = \oint(\mathrm{d}t/\mathrm{d}q)\mathrm{d}q = \oint(m/p)\mathrm{d}q = \oint m\ \mathrm{d}q/\sqrt{2m(E - U(q))}$,

$$T = \frac{\mathrm{d}}{\mathrm{d}E}\oint\sqrt{2m(E - U(q))}\,\mathrm{d}q = \frac{\mathrm{d}}{\mathrm{d}E}\oint p\ \mathrm{d}q = \frac{\mathrm{d}A}{\mathrm{d}E} \quad , \tag{1.10}$$

where A is the area enclosed.