1.3 Flow in phase space

Consider a region in the (p, q) plane and assume that each point in the region represents a possible position and momentum of the particle at the time $t = 0$. If each point moves according to the dynamics of the particle, those points will in general cover a different region at a later time $t > 0$. We can visualize this as the flow of a "fluid" in two dimensions.

If the forces are conservative, the fluid is incompressible. (See problem 1.5 for a dissipative case.)

For example, for the vertical motion of particles under gravity one has $q = (gt^2/2) + (p_0 t/m) + q_0$, $p = mgt + p_0$. The points of the rectangular region $0 \le q \le Q$, $0 \le p \le P$ at time $t = 0$, at a later time $t > 0$ will cover the parallelogram with vertices $a = (gt^2/2, mgt)$, $b = (gt^2/2 + Q, mgt)$, $c = (gt^2/2 + Pt/m + Q, P + mgt)$, $d = (gt^2/2 + Pt/m, P + mgt)$ (see figure 1.5). It can be seen by inspection that the area of this parallelogram is equal to the area PQ of the rectangle at time $t = 0$.

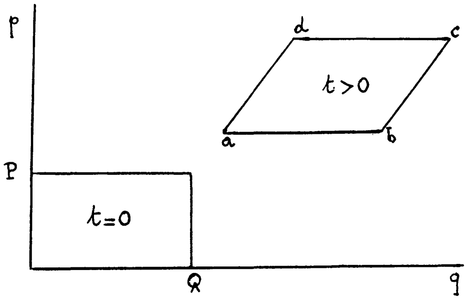

Figure 1.5: Vertical motion under gravity

As a prelude to the general case, we can also express the area at time $t > 0$ in the form

$$A_t = \int_{D_t} \mathrm{d}p_t \, \mathrm{d}q_t = \int_{D_0} \left| \frac{\partial(p_t, q_t)}{\partial(p_0, q_0)} \right| \mathrm{d}p_0 \, \mathrm{d}q_0 \ . \tag{1.11}$$

But the Jacobian determinant

$$\frac{\partial(p_t, q_t)}{\partial(p_0, q_0)} = \begin{vmatrix} \partial p_t/\partial p_0 & \partial p_t/\partial q_0 \\ \partial q_t/\partial p_0 & \partial q_t/\partial q_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ t/m & 1 \end{vmatrix} = 1$$

and so

$$A_t = \int_{D_0} \mathrm{d}p_0 \, \mathrm{d}q_0 = A_0 \ . \tag{1.12}$$

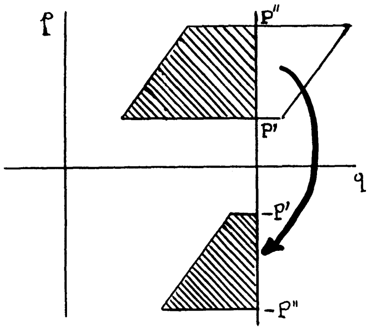

Figure 1.6: Elastic collision against a wall

The same property applies to particles undergoing elastic collision against a "wall" (see figure 1.6).

For the harmonic oscillator $q_t = q_0 \cos(\omega t) + p_0 \sin(\omega t)/m\omega$, $p_t = -m\omega q_0 \sin(\omega t) + p_0 \cos(\omega t)$ one has similarly

$$\frac{\partial(p_t, q_t)}{\partial(p_0, q_0)} = \begin{vmatrix} \cos(\omega t) & -m\omega \sin(\omega t) \\ \sin(\omega t)/m\omega & \cos(\omega t) \end{vmatrix} = 1 \ ,$$

and so the area conservation applies also in this case.

Can this property be generalized to any conservative force? Since $p_{t+\mathrm{d}t} = p_t - (\mathrm{d}U/\mathrm{d}q)_t \, \mathrm{d}t$, $q_{t+\mathrm{d}t} = q_t + (p_t/m)\, \mathrm{d}t$, one has

$$\frac{\partial(p_{t+\mathrm{d}t}, q_{t+\mathrm{d}t})}{\partial(p_t, q_t)} = \begin{vmatrix} 1 & -(\mathrm{d}^2U/\mathrm{d}q^2)_t \mathrm{d}t \\ \mathrm{d}t/m & 1 \end{vmatrix} = 1 + O((\mathrm{d}t)^2) \ .$$

Hence

$$\frac{\mathrm{d}A_t}{\mathrm{d}t} = 0 \ . \tag{1.13}$$

This is a special case of Liouville's theorem, which will be proved in Chapter 7.

For a one-dimensional system with conservative forces, by Stokes' theorem area conservation in the (p, q) plane is equivalent to conservation of a line integral along the curve enclosing the area,

$$\oint_{C_t} p \ \mathrm{d}q = \oint_{C_0} p \ \mathrm{d}q \ , \tag{1.14}$$

where C_0 and C_t are the boundaries of D_0 and D_t, respectively.

Direct proof: If $p = p(\lambda, t)$, $q = q(\lambda, t)$ are the parametric equations of C_t, where the domain of λ does not depend on t, one has

$$\oint_{C_t} p \ \mathrm{d}q = \oint \mathrm{d}\lambda \ p(\lambda, t) \ \partial q(\lambda, t)/\partial\lambda \ ,$$

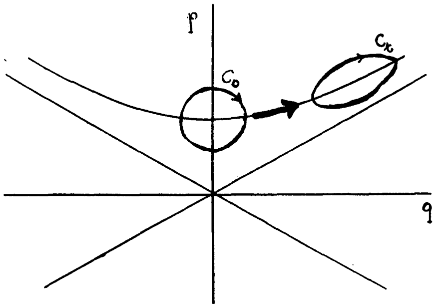

Figure 1.7: Example of equation (1.14)

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{C_t} p\ \mathrm{d}q = \oint\left(\frac{\partial p}{\partial t}\frac{\partial q}{\partial\lambda} + p\frac{\partial^2 q}{\partial\lambda\partial t}\right)\mathrm{d}\lambda = \oint\left(\frac{\partial p}{\partial t}\frac{\partial q}{\partial\lambda} - \frac{\partial p}{\partial\lambda}\frac{\partial q}{\partial t}\right)\mathrm{d}\lambda$$

$$= \oint\left[-\frac{\partial U}{\partial q}\frac{\partial q}{\partial\lambda} - \frac{\partial}{\partial\lambda}\left(\frac{p^2}{2m}\right)\right]\mathrm{d}\lambda = -\oint\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{p^2}{2m}+U\right)\mathrm{d}\lambda = 0 \quad .$$

As an example, let C_0 be the circle of equations

$$q(\lambda) = r\ \cos\lambda,\ p(\lambda) = p_0 - r\ \sin\lambda \ ,$$

and C_t the curve

$$p(\lambda,t) = m\omega q(\lambda)\ \sinh(\omega t) + p(\lambda)\ \cosh(\omega t) \ ,$$
$$q(\lambda,t) = q(\lambda)\ \cosh(\omega t) + (p(\lambda)/m\omega)\ \sinh(\omega t) \ ,$$

which is the mapping of C_0 under the flow induced by the repulsive force $f(q) = kq$ ($k = m\omega^2$). One has (see figure 1.7)

$$\oint_{C_t} p\ \mathrm{d}q = \oint_0^{2\pi}[m\omega r\ \cos\lambda\ \sinh(\omega t) + (p_0 - r\ \sin\lambda)\cosh\ (\omega t)]$$
$$[-r\ \sin\lambda\ \cosh(\omega t) - r\ \cos\lambda\ \sinh(\omega t)/m\omega]\ \mathrm{d}\lambda$$
$$= -r^2\pi[\sinh^2(\omega t) - \cosh^2(\omega t)] = +\pi r^2 = \oint_{C_0} p\ \mathrm{d}q \quad .$$

If C_0 is a (p,q)-orbit of a periodic motion, then $C_t = C_0$ at all times. In the phase flow, the points of C_0 pursue one another along C_0 itself. In this case,

$$J = \oint_{C_0} p\ \mathrm{d}q \tag{1.15}$$

is called a "cyclic action variable".

For the harmonic oscillator

$$q(\lambda,t) = a\ \cos(\omega t+\lambda),\ p(\lambda,t) = -m\omega a\ \sin(\omega t+\lambda),$$

$$J = \oint m\omega^2 a^2\ \sin^2(\omega t+\lambda)\ \mathrm{d}t = m\omega a^2\int_0^{2\pi}\sin^2\theta\ \mathrm{d}\theta = 2\pi\frac{E}{\omega} \quad . \tag{1.16}$$

The energy can be expressed in terms of J,

$$E = \frac{\omega J}{2\pi} \quad . \tag{1.17}$$

Since $J = A$, this agrees with the relation $A = 2\pi E\sqrt{m/k} = 2\pi E/\omega$ on page 4.

In the "old quantum mechanics", Sommerfeld's quantization condition

$$J = nh \qquad (n = 0, 1 \ldots) \tag{1.18}$$

yielded the energy levels for the harmonic oscillator

$$E = n\hbar\omega \tag{1.19}$$

instead of the correct $E = (n + 1/2)\hbar\omega$.

For a multiply periodic system with n degrees of freedom, the energy can be expressed as a function of n cyclic action variables, $E = E(J_1, J_2 \ldots J_n)$. An example will be presented in the next chapter (Keplerian motion) and the general case will be discussed in chapter 8.

1.4 The action integral

The cyclic action variable J originates from Hamilton's "characteristic function"

$$S(q_2, q_1, E) = \int_{q_1}^{q_2} p \, \mathrm{d}q \quad . \tag{1.20}$$

This is a very useful function. We notice first that the momenta at q_1 and q_2 are given by $p_2 = \partial S/\partial q_2$ and $p_1 = -\partial S/\partial q_1$.

The time between two positions q_1 and q_2 is given by

$$t_2 - t_1 = \partial S(q_2, q_1, E)/\partial E \tag{1.21}$$

In fact

$$t_2 - t_1 = \int_{q_1}^{q_2} \frac{\mathrm{d}t}{\mathrm{d}q}\mathrm{d}q = m\int_{q_1}^{q_2} \frac{\mathrm{d}q}{p} = m\int_{q_1}^{q_2} \frac{\mathrm{d}q}{\sqrt{2m(E - U(q))}}$$

$$= \frac{\partial}{\partial E}\int_{q_1}^{q_2} \sqrt{2m(E - U(q))} \, \mathrm{d}q = \frac{\partial}{\partial E} S(q_2, q_1, E) \quad .$$

The formula $T = \mathrm{d}J/\mathrm{d}E$ for the period of a closed orbit is a special case for $q_2 = q_1$.