The energy can be expressed in terms of J,

$$E = \frac{\omega J}{2\pi} \quad . \tag{1.17}$$

Since $J = A$, this agrees with the relation $A = 2\pi E\sqrt{m/k} = 2\pi E/\omega$ on page 4.

In the "old quantum mechanics", Sommerfeld's quantization condition

$$J = nh \qquad (n = 0, 1 \ldots) \tag{1.18}$$

yielded the energy levels for the harmonic oscillator

$$E = n\hbar\omega \tag{1.19}$$

instead of the correct $E = (n + 1/2)\hbar\omega$.

For a multiply periodic system with n degrees of freedom, the energy can be expressed as a function of n cyclic action variables, $E = E(J_1, J_2 \ldots J_n)$. An example will be presented in the next chapter (Keplerian motion) and the general case will be discussed in chapter 8.

1.4 The action integral

The cyclic action variable J originates from Hamilton's "characteristic function"

$$S(q_2, q_1, E) = \int_{q_1}^{q_2} p \, \mathrm{d}q \quad . \tag{1.20}$$

This is a very useful function. We notice first that the momenta at q_1 and q_2 are given by $p_2 = \partial S/\partial q_2$ and $p_1 = -\partial S/\partial q_1$.

The time between two positions q_1 and q_2 is given by

$$t_2 - t_1 = \partial S(q_2, q_1, E)/\partial E \tag{1.21}$$

In fact

$$t_2 - t_1 = \int_{q_1}^{q_2} \frac{\mathrm{d}t}{\mathrm{d}q}\mathrm{d}q = m\int_{q_1}^{q_2} \frac{\mathrm{d}q}{p} = m\int_{q_1}^{q_2} \frac{\mathrm{d}q}{\sqrt{2m(E - U(q))}}$$

$$= \frac{\partial}{\partial E}\int_{q_1}^{q_2} \sqrt{2m(E - U(q))}\, \mathrm{d}q = \frac{\partial}{\partial E} S(q_2, q_1, E) \quad .$$

The formula $T = \mathrm{d}J/\mathrm{d}E$ for the period of a closed orbit is a special case for $q_2 = q_1$.

For the harmonic oscillator one has

$$S(q_2, q_1, E = ka^2/2) = \int_{q_1}^{q_2} \sqrt{2m(E - kq^2/2)}\, \mathrm{d}q = m\omega \int_{q_1}^{q_2} \sqrt{a^2 - q^2}\, \mathrm{d}q$$

$$= (m\omega/2)\left[q_2\sqrt{a^2 - q_2^2} + a^2 \sin^{-1}(q_2/a) - \text{same for } q_1\right] \quad , \tag{1.22}$$

while (1.21) gives

$$t_2 - t_1 = [\sin^{-1}(q_2/a) - \sin^{-1}(q_1/a)]/\omega \quad . \tag{1.23}$$

Taking $q_1 = a$, $t_1 = t_0$, $q_2 = q$, $t_2 = t$, we have

$$t - t_0 = [\sin^{-1}(q/a) - \pi/2]/\omega \ , \ q = a\cos(\omega(t - t_0)) \quad . \tag{1.24}$$

The action $S(q, q_0, E)$ obeys the Hamilton-Jacobi equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + U(q) = E \quad , \tag{1.25}$$

as can be seen by substituting $p = \partial S/\partial q$ in the expression for the energy, $E = p^2/2m + U(q)$.

As an example, verify that the action for a particle of positive energy E subject to the force $f(q) = +kq$, $U = -kq^2/2$, is

$$S(q, q_0, E) = \int_{q_0}^{q} \sqrt{2m(E - U(q'))}\, \mathrm{d}q'$$

$$= \frac{m\omega}{2}\left[q\sqrt{q^2 + 2E/k} - q_0\sqrt{q_0^2 + 2E/k} + \frac{2E}{k}\ln\left(\frac{q + \sqrt{q^2 + 2E/k}}{q_0 + \sqrt{q_0^2 + 2E/k}}\right)\right] \quad , \tag{1.26}$$

and that it satisfies the Hamilton-Jacobi equation for $U(q) = -kq^2/2$.

All this may seem somewhat futile. However, the usefulness of action integrals becomes evident as soon as we extend their definition to more than one dimension, say three dimensions.

Let

$$S(\mathbf{r}, \mathbf{r}_0, E, \alpha) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{p} \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}} \sqrt{2m(E - U(\mathbf{r}))}\, \mathrm{d}s \quad , \tag{1.27}$$

where the line integral is along a trajectory characterized by the energy E and some integrals of motion summarily referred to by the symbol α. (In one dimension, the only integral of motion was E, hence no α.) Note that along a trajectory $\mathbf{p} \cdot \mathrm{d}\mathbf{r} = |\mathbf{p}|\, \mathrm{d}s$, $\mathrm{d}s = |\mathrm{d}\mathbf{r}|$.

For the two-dimensional case of projectile motion, with the x-axis horizontal and the y-axis vertically upwards, and $\mathbf{r}_0 = 0$, in the ascending branch of the trajectory one finds

$$S(\mathbf{r}, 0, E, \alpha = p_x) = \int_0^x p_x \, \mathrm{d}x' + \int_0^y \sqrt{2m(E - E_x - mgy')} \, \mathrm{d}y'$$

$$= p_x x + [(2m(E - E_x))^{\frac{3}{2}} - (2m(E - E_x - mgy))^{\frac{3}{2}}]/3m^2 g \quad , \qquad (1.28)$$

where $E_x = p_x^2/2m$.

Consider the normal to the curve $S = 0$ at $\mathbf{r} = 0$. Since $\partial S/\partial x = p_x = mv_{0x}$ and $(\partial S/\partial y)_{y=0} = \sqrt{2m(E - E_x)} = mv_{0y}$, it is clear that the trajectory is normal to the curve $S = 0$ at the $\mathbf{r} = 0$ intersection.

This is not surprising. In the three-dimensional case, the normal to the surface $S(\mathbf{r}, \mathbf{r}_0, E, \alpha) = 0$ at $\mathbf{r}_0$ is parallel to the gradient $(\nabla S)_{\mathbf{r}_0}$, and this in turn is equal to the momentum $\mathbf{p}$ at $\mathbf{r}_0$.

A method for finding trajectories suggests itself. The action integral satisfies the Hamilton-Jacobi equation

$$(\nabla S)^2/(2m) + U = E \quad . \qquad (1.29)$$

If $S(\mathbf{r})$ is a solution of this equation, consider the family of surfaces $S(\mathbf{r}) = \text{const}$. The trajectories form a family of curves orthogonal to these surfaces. The problem of finding trajectories is similar to that of finding lines of force in electrostatics once the equipotential surfaces are known.

Let us re-examine the problem of projectile motion from this viewpoint. The Hamilton-Jacobi equation

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy = E \qquad (1.30)$$

can be solved by separation of variables, $S(x, y) = S_x(x) + S_y(y)$, $(\mathrm{d}S_x/\mathrm{d}x)^2/(2m) = E_x$, $(\mathrm{d}S_y/\mathrm{d}y)^2/(2m) + mgy = E_y$, with $E_x + E_y = E$.

The function

$$S(x, y) = \sqrt{2mE_x}\, x - [(2m(E - E_x - mgy))^{\frac{3}{2}} - (2m(E - E_x))^{\frac{3}{2}}]/3m^2 g \qquad (1.31)$$

is a solution of equation (1.30), corresponding to a curve passing through the origin.

To find the trajectory through the origin corresponding to the energies E_x and E_y, we write

$$\mathrm{d}y/\mathrm{d}x = (\partial S/\partial y)/(\partial S/\partial x) = \sqrt{E - E_x - mgy}/\sqrt{E_x} \quad ,$$

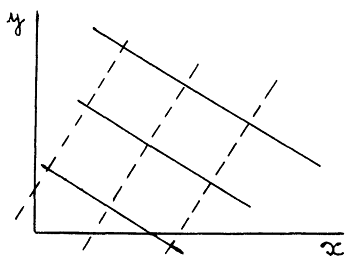

Figure 1.8: All trajectories in this figure correspond to the same values of p_x and p_y.

where we have chosen the positive sign for $\mathrm{d}y/\mathrm{d}x$ (ascending branch of the trajectory). This expresses the fact that a displacement $(\mathrm{d}x, \mathrm{d}y)$ along the trajectory is parallel to the normal to the curve represented by $S(x, y)$.

Separating variables, we have

$$\mathrm{d}y/\sqrt{E - E_x - mgy} = \mathrm{d}x/\sqrt{E_x} \quad ,$$

$$-(2/mg)\sqrt{E - E_x - mgy} = x/\sqrt{E_x} - (2/mg)\sqrt{E - E_x} \quad .$$

Putting $E_x = mv_{0x}^2/2$ and $E - E_x = mv_{0y}^2/2$, and assuming that v_{0x} and v_{0y} are both positive, this reduces to the familiar formula

$$y = \frac{v_{0y}}{v_{0x}}x - \frac{gx^2}{2v_{0x}^2} \quad . \tag{1.32}$$

This, of course, can be derived by equating the two expressions for t, x/v_{0x} and $(v_{0y} - \sqrt{v_{0y}^2 - 2gy}\,)/g$.

The Hamilton-Jacobi equation for a free particle of energy E in two dimensions with momentum component p_x has the solution

$$S = p_x x \pm \sqrt{2m(E - E_x)}\, y \quad . \tag{1.33}$$

The curves $S = \text{const}$ (solid) and the trajectories (broken) are shown in figure 1.8 for $p_x > 0$ and the plus sign. Of course, the roles of x and y can be interchanged.

However, the Hamilton-Jacobi equation for a free particle of energy E in two dimensions has also the solution

$$S = \pm\sqrt{2mE}\,|\mathbf{r} - \mathbf{r}_0| \quad . \tag{1.34}$$

The curves S=const are circles, the trajectories are straight lines through $\mathbf{r}_0$ (see figure 1.9). The angular momentum about $\mathbf{r}_0$, $l = 0$, plays the role of the constant of motion α other than the energy (see equation (1.27)).

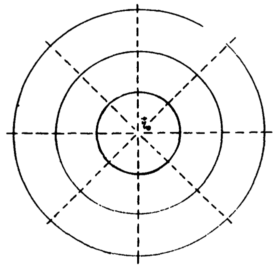

Figure 1.9: Equation (1.34)

1.5 The Maupertuis principle

We have seen that

$$S(B) - S(A) = \left(\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{p} \cdot \mathrm{d}\mathbf{r} \right)_{\mathrm{t}} = \left(\int_{\mathrm{A}}^{\mathrm{B}} |\mathbf{p}| \mathrm{d}s \right)_{\mathrm{t}} \quad , \tag{1.35}$$

where $\mathbf{p} = \nabla S$, $|\mathbf{p}| = |\nabla S| = \sqrt{2m(E - U(\mathbf{r})}$ and "t" is the actual trajectory from A to B.

Now if "c" is a curve from A to B adjacent to the trajectory, we see that

$$S(B) - S(A) = \left(\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{p} \cdot \mathrm{d}\mathbf{r} \right)_{\mathrm{c}} \leq \left(\int_{A}^{B} \sqrt{2m(E - U(\mathbf{r}))}\, \mathrm{d}s \right)_{\mathrm{c}} \quad , \tag{1.36}$$

since $\mathbf{p}$ and $\mathrm{d}\mathbf{r}$ are not parallel on "c".

Hence the Maupertuis principle or principle of "least action": The action integral from a point A to a point B, $\int_A^B \sqrt{2m(E - U(\mathbf{r})}\, \mathrm{d}s$, is minimal along a trajectory. Less restrictively, Euler's formulation of the principle states that

$$\delta \int_{A}^{B} \sqrt{E - U(\mathbf{r})}\, \mathrm{d}s = 0 \quad , \tag{1.37}$$

where δ indicates variation of the integral when the coordinates $\mathbf{r} = (x, y, z)$ of each point of a trajectory from A to B are changed to $\mathbf{r} + \delta\mathbf{r} = (x + \delta x, y + \delta y, z + \delta z)$, with $\delta\mathbf{r}$ vanishing at A and B.

It must be possible to derive the equations of trajectories from this variational principle. Let us do so in two dimensions and assuming that $U = U(y)$. With $y' \equiv \mathrm{d}y/\mathrm{d}x$, we have

$$\delta \int_{A}^{B} \sqrt{E - U} \mathrm{d}s = \delta \int_{A}^{B} \sqrt{E - U(y)}\, \sqrt{1 + y'^2}\, \mathrm{d}x$$