

$$= \int_A^B \left[-\frac{1}{2} \sqrt{\frac{1+y'^2}{E-U}} \frac{dU}{dy} \delta y + \sqrt{\frac{E-U}{1+y'^2}} y' \frac{d\delta y}{dx} \right] dx = 0 .$$

Partial integration using $\delta y_A = \delta y_B = 0$ yields the trajectory equation

$$\frac{d}{dx} \left(y' \sqrt{\frac{E-U}{1+y'^2}} \right) + \frac{1}{2} \sqrt{\frac{1+y'^2}{E-U}} \frac{dU}{dy} = 0 .$$

Multiplying by $y' \sqrt{(E-U)/(1+y'^2)}$ we obtain

$$\frac{d}{dx} \left(\frac{y'^2}{1+y'^2} (E-U) + U \right) = 0 ,$$

so that the quantity in parentheses is a constant, which we denote by C . Hence

$$y'^2 = \frac{C-U}{E-C} .$$

The value of the constant can be determined from that of y' at a point of the trajectory. For a projectile ($U = mgy$), we have $E = mv_0^2/2$, $(y')_{x=y=0} = v_{0y}/v_{0x}$, and so $C = mv_{0y}^2/2$, $y' = (\sqrt{v_{0y}^2 - 2gy})/v_{0x}$. Solution by separation of variables yields equation (1.32) if the value of the integration constant is chosen so that the trajectory may pass through the origin.

1.6 The time

What about the time? Equation (1.21) expressed the time interval during which a particle moves from a position to another as the derivative of the action integral with respect to the energy. That formula is easily generalized to three dimensions. In fact

$$\begin{aligned} \frac{\partial}{\partial E} S(\mathbf{r}_2, \mathbf{r}_1, E, \alpha) &= \frac{\partial}{\partial E} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{p} \cdot d\mathbf{r} = \frac{\partial}{\partial E} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sqrt{2m(E-U(\mathbf{r}))} ds \\ &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} m ds / \sqrt{2m(E-U(\mathbf{r}))} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} ds/v = t_2 - t_1 , \end{aligned} \quad (1.38)$$

where v denotes the velocity.

It is interesting to compare this time interval with the time it would take a particle of the same mass, total energy E , and potential energy $U(\mathbf{r})$, to

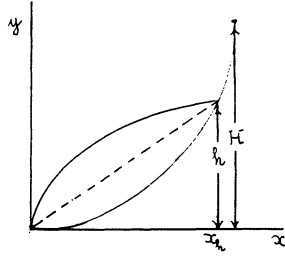


Figure 1.10: Real and varied motion under gravity

go from \mathbf{r}_1 to \mathbf{r}_2 along a path other than the trajectory. This latter motion requires a frictionless constraint.

Consider as an example the motion of a projectile. Since

$$S(x, y, 0, E, p_x = mv_{0x}) = p_x x + [(2E - mv_{0x}^2)^{\frac{3}{2}} - (2E - mv_{0x}^2 - 2mgy)^{\frac{3}{2}}] / 3\sqrt{m}g \quad ,$$

the time t from $(0, 0)$ to (x, y) is

$$t = \partial S / \partial E = \left[\sqrt{2E - mv_{0x}^2} - \sqrt{2E - mv_{0x}^2 - 2mgy} \right] / g\sqrt{m} \\ = \left(v_{0y} - \sqrt{v_{0y}^2 - 2gy} \right) / g \quad ,$$

a formula known to high-school students. The time taken to reach the highest point of the trajectory, $y = h = v_{0y}^2 / 2g$, $x = x_h = v_{0x}v_{0y} / g$, is, of course, $t = v_{0y} / g$.

Note that instead of choosing p_x as a constant of motion α other than the total energy E , we might have chosen the energy of the y -motion, $E_y = mv_y^2 / 2 + mgy$. Then we would have had

$$t = \partial S_x / \partial E = \partial (\sqrt{2m(E - E_y)} x) / \partial E = x \sqrt{m / 2(E - E_y)} = x / v_{0x}.$$

Suppose now that the body, with the same total energy $E = m(v_{0x}^2 + v_{0y}^2) / 2$, travels from $(0, 0)$ to (x_h, h) along the straight line of equation $y = hx / x_h$ (see figure 1.10) under the influence of gravity and a frictionless constraint.

The time t' taken will be shorter than $t = v_{0y} / g$,

$$t' = \int m ds / \sqrt{2m(E - mgy)} = \int_0^h dy \sqrt{1 + (x_h/h)^2} / \sqrt{v_0^2 - 2gy}$$

$$= (1/g)\sqrt{1 + 4v_{0x}^2/v_{0y}^2} \left(\sqrt{v_{0x}^2 + v_{0y}^2} - v_{0x} \right) < v_{0y}/g = t \quad .$$

Does a shortest-time curve from $(0,0)$ to (x_h, h) exist for that energy E ? Yes, it is Bernoulli's "brachistochrone".

Its equation is obtained by requiring that

$$\delta \int_A^B \frac{ds}{v} = 0 \quad (1.39)$$

for variations $\delta \mathbf{r}$ restricted to vanish at A and B .

A calculation similar to that presented in section 1.5 yields the differential equation

$$\frac{d}{dx} \left(\frac{y'^2}{(E-U)(1+y'^2)} \right) = \frac{1}{(E-U)^2} \frac{\partial U}{\partial y} y' \quad .$$

For $U = mgy$ one obtains by a first integration

$$y' = \sqrt{\frac{C - (H - y)}{H - y}} \quad ,$$

where C is an integration constant and $H = E/mg$. Note that y' is infinite (tangent is vertical) for $y = H$, indicating that $v_x = 0$ for $y = H$. Thus H is indeed the maximum height reached by a body of energy E .

A second integration putting $H - y = C(1 + \cos\theta)/2$, and a suitable choice of integration constants, yield

$$x = H(\theta + \sin\theta)/2 \quad , \quad y = H(1 - \cos\theta)/2 \quad . \quad (1.40)$$

These are the parametric equations of a cycloid. Thus the brachistochrone is also "tautochrone" (see problem 1.8).

Note that $x = y = 0$ for $\theta = 0$, and $(x = \pi H/2, y = H)$ for $\theta = \pi$. The time taken by the projectile of energy $E = mgH$ to go from $(0,0)$ to the point characterized by the value θ of the parameter is

$$\begin{aligned} t'' &= \int \frac{ds}{v} = \sqrt{\frac{H}{2g}} \int_0^y \frac{dy}{\sqrt{y(H-y)}} \\ &= \sqrt{\frac{H}{2g}} \int_0^\theta \frac{\sin\theta \, d\theta}{\sqrt{(1+\cos\theta)(1-\cos\theta)}} = \sqrt{\frac{H}{2g}} \theta \quad , \end{aligned}$$

where we have used $ds = \sqrt{1 + (dx/dy)^2} dy = \sqrt{H/y} dy$, $v = \sqrt{2g(H-y)}$, and the parametric equations.

We now want to find a brachistochrone through the points $(0, 0)$ and (x_h, h) of the projectile trajectory. It is easy to show that a brachistochrone for $E = mv_0^2/2 = m(v_{0x}^2 + v_{0y}^2)/2$ and $H = v_0^2/2g > v_{0y}^2/2g = h$ will intersect the projectile trajectory at (x_h, h) for the value $\theta = 2v_{0x}v_{0y}/v_0^2$ of the parameter. The time from $(0, 0)$ to (x_h, h) will be

$$t'' = \frac{1}{\sqrt{1 + (v_{0y}/v_{0x})^2}} \frac{v_{0y}}{g} < \frac{v_{0y}}{g} = t \quad .$$

It is also easy to show that $t'' < t'$, see problem 1.9.

1.7 Fermat's principle

We must briefly mention the remarkable similarity between geometrical optics and mechanics of a point-mass. In the former, Fermat's principle

$$\delta \int_A^B \frac{n(\mathbf{r})}{c} ds = 0 \quad (1.41)$$

expresses the fact that the time taken by light to travel from A to B along a light ray ("optical path length") is less than the time it would take along any adjacent path from A to B .

In mechanics, Maupertuis' principle is the analogue of Fermat's principle. The action (not the time) is the analogue of the optical path length.

Fermat's principle is the consequence of the existence of families of "iconal surfaces" to which the families of light rays are orthogonal, in the same way as trajectories are orthogonal to $S = \text{const}$ surfaces.

An "iconal equation" is obtained as first approximation of the wave equation for a given frequency ν ,

$$\nabla^2 \psi + \frac{4\pi^2}{\lambda^2} \psi = 0 \quad , \quad (1.42)$$

where $\lambda = \lambda_0/n = c/n\nu$, λ_0 is the vacuum wavelength, and $n(\mathbf{r})$ is the refractive index.

One begins by expressing ψ in the form

$$\psi = \exp(2\pi i S(\mathbf{r})/\lambda_0) \quad , \quad (1.43)$$

where $S = S_0 + (\lambda_0/2\pi)S_1 + (\lambda_0/2\pi)^2 S_2 + \dots$

Substituting in the wave equation, one finds in first approximation the iconal equation

$$(\nabla S_0)^2 = n^2 \quad . \quad (1.44)$$