We now want to find a brachistochrone through the points $(0,0)$ and (x_h, h) of the projectile trajectory. It is easy to show that a brachistochrone for $E = mv_0^2/2 = m(v_{0x}^2 + v_{0y}^2)/2$ and $H = v_0^2/2g > v_{0y}^2/2g = h$ will intersect the projectile trajectory at (x_h, h) for the value $\theta = 2v_{0x}v_{0y}/v_0^2$ of the parameter. The time from $(0,0)$ to (x_h, h) will be

$$t'' = \frac{1}{\sqrt{1+(v_{0y}/v_{0x})^2}}\frac{v_{0y}}{g} < \frac{v_{0y}}{g} = t \quad .$$

It is also easy to show that $t'' < t'$, see problem 1.9.

1.7 Fermat's principle

We must briefly mention the remarkable similarity between geometrical optics and mechanics of a point-mass. In the former, Fermat's principle

$$\delta \int_A^B \frac{n(\mathbf{r})}{c}\, \mathrm{d}s = 0 \tag{1.41}$$

expresses the fact that the time taken by light to travel from A to B along a light ray ("optical path length") is less than the time it would take along any adjacent path from A to B.

In mechanics, Maupertuis' principle is the analogue of Fermat's principle. The action (not the time) is the analogue of the optical path length.

Fermat's principle is the consequence of the existence of families of "iconal surfaces" to which the families of light rays are orthogonal, in the same way as trajectories are orthogonal to $S =$const surfaces.

An "iconal equation" is obtained as first approximation of the wave equation for a given frequency ν,

$$\nabla^2\psi + \frac{4\pi^2}{\lambda^2}\psi = 0 \quad , \tag{1.42}$$

where $\lambda = \lambda_0/n = c/n\nu$, λ_0 is the vacuum wavelength, and $n(\mathbf{r})$ is the refractive index.

One begins by expressing ψ in the form

$$\psi = \exp(2\pi \mathrm{i} S(\mathbf{r})/\lambda_0) \quad , \tag{1.43}$$

where $S = S_0 + (\lambda_0/2\pi)S_1 + (\lambda_0/2\pi)^2 S_2 + \ldots$

Substituting in the wave equation, one finds in first approximation the iconal equation

$$(\nabla S_0)^2 = n^2 \quad . \tag{1.44}$$

This is the analogue of the Hamilton-Jacobi equation.

Wave mechanics (de Broglie) is to classical mechanics what wave optics is to geometrical optics. In fact, comparing the Schrödinger equation

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-U)\psi = 0 \tag{1.45}$$

with equation (1.42) one sees that $\sqrt{2m(E-U)}$ plays the role of a refractive index, while the wave-mechanical wavelength is

$$\lambda = h/\sqrt{2m(E-U)} = h/p(\mathbf{r}).$$