## 1.8 Chapter 1 problems

**1.1** The damped oscillator equation $\ddot{q} = -q - \dot{q}$ is equivalent to the system $\dot{q} = p$, $\dot{p} = -q - \gamma p$. Show that these equations cannot be expressed in the Hamiltonian form (1.3-4).

**1.2** The damped oscillator equation of the preceding problem is also equivalent to $\dot{q} = p\exp(-\gamma t)$, $\dot{p} = -q\ \exp(\gamma t)$. Verify that these equations are Hamilton equations, with $H = [p^2\exp(-\gamma t) + q^2\exp(\gamma t)]/2$.

**1.3** The equation $\ddot{q} = -(q - q_0)$ for a harmonic oscillator with equilibrium point $q = q_0$, can be converted into the system of equations $\dot{q} = p - p_0$, $\dot{p} = -q + q_0$, where $p_0$ is an arbitrary constant. These are Hamilton equations, since $\partial(p - p_0)/\partial q = 0$ and $\partial(-q + q_0)/\partial p = 0$. (i) Find the Hamiltonian. (ii) What is the $(p, q)$ trajectory for energy $E$?

**1.4** Show that the 4-dimensional phase space volume is conserved in the elastic collision of two particles in one dimension.

**1.5** Non-conservative forces invalidate Liouville's theorem. Show that, if $\dot{p} = -\gamma\dot{q}$ $(\gamma > 0)$, $\dot{q} = p/m$, areas in the $(p, q)$-plane decrease exponentially with time.

**1.6** According to Liouville's theorem, conservative forces cannot change the small-scale phase space density of a system of particles. Yet they can change the large-scale density producing accumulation.

Give a simple one-dimensional example (two-dimesnional phase space). Look up also S. van der Meer, "Stochastic cooling and the accumulation of antiprotons", *Revs. Mod. Phys.*,**57**(1985)689.

**1.7** For the two-dimensional oscillator with $T = m(\dot{x}^2 + \dot{y}^2)/2$, $U = m\omega^2(x^2 + y^2)/2$, and energy $E = E_x + E_y$ with $E_x = m\omega^2 a^2/2$, $E_y = m\omega^2 b^2/2$, the action integral is given by $S = S_x + S_y$, with $S_x$ obtained from equation (1.22) with the replacement $q_2 \to x$, $q_1 \to x_0$ and $S_y$ with the replacements $a \to b$, $q_2 \to y$, $q_1 \to y_0$.

Show that

$$t - t_0 = [\sin^{-1}(x/a) - \sin^{-1}(x_0/a)]/\omega = [\sin^{-1}(y/b) - \sin^{-1}(y_0/b)]/\omega \quad .$$

Equating these two expressions for $t - t_0$, show that the trajectory equation is

$$b^2x^2 + a^2y^2 - 2ab\cos(\alpha - \beta)xy = a^2b^2\sin^2(\alpha - \beta) \quad \text{(an ellipse)} \quad ,$$

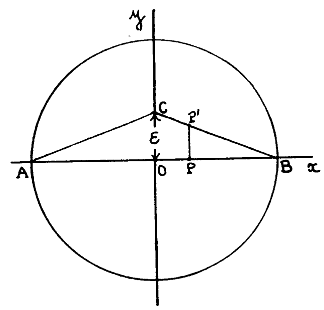


Figure 1.11: Illustration of Maupertuis principle

where $\alpha = \cos^{-1}(x_0/a)$ and $\beta = \cos^{-1}(y_0/b)$.

**1.8** A particle, initially at rest, slides down without friction along the brachistochrone (1.40). Show that the time required to reach the bottom is $\pi\sqrt{H/2g}$ , independent of the initial position. Thus the brachistochrone is also tautochrone.

**1.9** With the notation of section 1.6, show that $t'' < t'$.

**1.10** In figure 1.11, the circle represents an earth meridian, $A$ and $B$ are the initial and final positions of an object released from rest at $A$ and travelling to $B$ along a smooth straight tunnel.

Assuming $\epsilon$ infinitesimal, show that

$$D = \left(\int \sqrt{2m(E - U(P'))}\, \mathrm{d}s'\right)_{ACB} - \left(\int \sqrt{2m(E - U(P))}\, \mathrm{d}s\right)_{AOB} > 0$$

as expected from the Maupertuis principle.

## Solutions to ch. 1 problems

**S1.1** $\dot{p} = -\partial H/\partial q$, $\dot{q} = \partial H/\partial p$ would give $-q - \gamma p = -\partial H/\partial q$, $p = \partial H/\partial p$. Differentiating the first with respect to $p$ we have $\partial^2 H/\partial p\partial q = \gamma$, while differentiating the second with respect to $q$ we have $\partial^2 H/\partial q\partial p = 0$.

**S1.3** (i) $H = [(p-p_0)^2 + (q-q_0)^2]/2$ (ii) $q = q_0 + \sqrt{2E}\,\cos t$, $p = p_0 - \sqrt{2E}\,\sin t$, circle of radius $\sqrt{2E}$ and center $(p_0, q_0)$.

**S1.4** $Mp_1 = (m_1 - m_2)p_{10} + 2m_1p_{20}$, $Mp_2 = 2m_2p_{10} + (m_2 - m_1)p_{20}$,
$M = m_1 + m_2$
$\int \mathrm{d}p_1\mathrm{d}q_1\mathrm{d}p_2\mathrm{d}q_2 = \int \mathrm{d}q_{10}\mathrm{d}q_{20} \int |\partial(p_1,p_2)/\partial(p_{10},p_{20})| \mathrm{d}p_{10}\mathrm{d}p_{20} = \int \mathrm{d}p_{10}\mathrm{d}q_{10}\mathrm{d}p_{20}\mathrm{d}q_{20}$,
since

$$\frac{\partial(p_1,p_2)}{\partial(p_{10},p_{20})} = \begin{vmatrix} M^{-1}(m_1-m_2) & 2M^{-1}m_1 \\ 2M^{-1}m_2 & M^{-1}(m_2-m_1) \end{vmatrix} = -1$$

**S1.5**

$$\frac{\partial(p_{t+\mathrm{d}t}, q_{t+\mathrm{d}t})}{\partial(p_t, q_t)} = \begin{vmatrix} 1 - m^{-1}\gamma\mathrm{d}t & 0 \\ m^{-1}\mathrm{d}t & 1 \end{vmatrix} = 1 - \frac{\gamma}{m}\,\mathrm{d}t \quad ,$$

and so $\mathrm{d}A_t/\mathrm{d}t = -m^{-1}\gamma A_t$, $A_t = A_0 \exp(-m^{-1}\gamma t)$.

**S1.6** In figure 1.2, think of two small phase space regions around $(q = a, p = 0)$ and $(q = -a, p = 0)$ filled with particles. If the $p$-axis of the ellipse is much smaller than the $q$-axis ($k$ very small), after a quarter of a period the two regions will have moved to $(q = 0, p = \pm m\omega a)$ with resulting accumulation around $(q = 0, p = 0)$.

In an antiproton accumulator the empty (phase) spaces between the particles are squeezed outwards, while each antiproton is pushed towards the center of the distribution.

**S1.7** Regarding $E_y$ (and, therefore, $b$) as constant of motion other than $E$, since $a = \omega^{-1}\sqrt{2(E - E_y)/m}$ , the first of the two expressions for $t - t_0$ is given by $t - t_0 = \partial S_x/\partial E = (\partial S_x/\partial a)(\partial a/\partial E)$. Also $t - t_0 = \partial S_y/\partial E = (\partial S_y/\partial b)(\partial b/\partial E)$.

The trajectory equation is obtained by equating the two expressions for $t - t_0$. This gives $\sin^{-1}(x/a) - \sin^{-1}(y/b) = \sin^{-1}(x_0/a) - \sin^{-1}(y_0/b)$, which yields the desired result by using the identity

$$\sin^{-1}\epsilon - \sin^{-1}\eta = \sin^{-1}\left(\epsilon\eta + \sqrt{(1-\epsilon^2)(1-\eta^2)}\right).$$

**S1.8** It is convenient to use the parametric equations of the cycloid in terms of the arc length $s$ from the lowest point, $x = \ldots$, $y = s^2/4H$. Then

$$m\ddot{s} = -mg\,\mathrm{d}y/\mathrm{d}s = -mgs/2H,\ \ddot{s} + \omega^2 s = 0 \text{ with } \omega = \sqrt{g/2H},$$

harmonic motion of the cycloidal pendulum.
The time sought is $T/4 = \pi/2\omega = \pi\sqrt{H/2g}$.

**S1.9** With $x = v_{0y}/v_{0x} > 0$, $t''?t'$ is equivalent to

$$\frac{1}{\sqrt{1+x^2}} ? \sqrt{1+\frac{4}{x^2}}\left(\sqrt{1+\frac{1}{x^2}}-\frac{1}{x}\right) \quad .$$

By simple algebra this yields $0?x^8 + 8x^6 + 20x^4 + 12x^2$. Thus "?" is "<".

**S1.10**

$$P(x,0),\ P'(x,\delta y) \text{ with } \delta y = \epsilon\eta(x+\eta R)/R,\ \eta = -x/|x|,$$
$$U(P) = mg(x^2 - 3R^2)/2R,\ U(P') = mg(x^2 + (\delta y)^2 - 3R^2)/2R,$$
$$E = -mgR,\ \sqrt{E-U(P')} - \sqrt{E-U(P)} = -mg(\delta y)^2/(4R\sqrt{E-U(P)}\ ),$$
$$\mathrm{d}s' - \mathrm{d}s \simeq \epsilon^2 \mathrm{d}x/(2R^2),$$

$$D \simeq \int_{-R}^{R} [\sqrt{2m(E-U(P'))} - \sqrt{2m(E-U(P))} + \frac{\epsilon^2}{2R^2}\sqrt{2m(E-U(P))}\ ]\,\mathrm{d}x$$

$$\simeq -\frac{m^{\frac{3}{2}} g \epsilon^2}{\sqrt{2}R^3}\int_{-R}^{R}\frac{x(x+\eta R)\mathrm{d}x}{\sqrt{E-U(P)}} > 0 \quad ,$$

since $x(x+\eta R) < 0$ in the integration interval.