PREFACE

Tyger! Tyger! burning bright
In the forests of the night,
What immortal hand or eye
Could frame thy fearful symmetry?

William Blake

The science of patterns should begin by trying to give a precise definition of pattern, but this seems neither feasible nor desirable. It is not feasible because patterns permeate human perception to such an extent that an objective definition seems unlikely. It is not desirable because any precise definition would only serve to limit ourselves unnecessarily. However, we will still try to address nontrivial questions:

Do patterns reflect the reality of the world? Patterns are inherent to our mental processes so it is not even clear whether they have an existence independent of our own perception—to what extent are we under collective hypnosis, tricked by a visual system eager to see Pegasus and Pisces in the Poisson distribution of stars?

By simply opening our eyes we are overwhelmed by the presence of patterns, so we can avoid overloading the brain by closing them and looking at the universe through the eye of a physicist. At the largest scales we see Lorentzian space–time gently curving according to 20 tensorial components and the Newtonian clockwork of stars and planets. At the smallest scales we see Hilbert spaces of quantum fields and 10 dimensions of superstrings. Here our analytical brain scores highest—mathematics, to the delight of Platonists, is able to capture the evanescent nature of matter.

Encouraged by such success we retain the framework of physics and look at patterns such as clouds in the sky, water flowing in a river, sand hills on a beach, and ice crystals on a frosty glass. We then try to describe what we see. Even in this limited context, human intellect is put to shame! We cannot precisely describe what we are perceiving, nor do we have a clear understanding of rules governing these common patterns. A satisfactory analysis based on the fundamental laws of physics and probability theory seems a long way off. These patterns can still be interpreted as physical phenomena, so their complexity might eventually be understood in terms of the deep mathematical interaction between local and global dynamics of matter. Yet the foliage of a tree, the concept of chair, or the face of a human being seem hopelessly beyond pure physical modeling.

Nevertheless, we can augment our model by using computational paradigms. In this way, pattern formation can be modeled as information flow in a computation while still retaining the local to global property of physical models [Prusinkiewicz].

"Local to global," a key geometric principle in representations of the world, is often expressed analytically in the language of partial differential equations (PDE's). Ubiquitous in physics, PDE's were imported into biology by Alan Turing. He envisioned how simultaneous diffusion of several interacting chemical substances might cause mosaic patterns on animals, from zebras and tigers to sea shells and fruit flies larvae [Meinhardt] [Argentina, Coullet].[1]

Such models possess symmetry, but what is actually observed is *broken symmetry,* in other words, partial symmetry (if any).[2] "Broken symmetry" does not evoke images of beauty, but aesthetically pleasing products of nature such as flowers can trace their allure to human sensibility to discrete symmetry. In fact, many flowers have finite but not full circular symmetry. According to Paul Green [Transduction to generate plant form and pattern: an essay on cause and effect, *Annals of Botany* **78**:269–281, 1996.] petals grow layer–by–layer with each consecutive layer taking on a symmetric arrangement determined by the lowest eigenfunction of an elasticity problem constrained by the previous stage of morphogenesis. Similar models were also used by Green to model phyllotaxis, the arrangement of leaves on a stem, of scales on a pine cone and of florets in a flower. Here an amazing regularity emerges and can lead to the appearance of Fibonacci numbers. Alternative models of phyllotaxis using finite nonlinear dimensional dynamics rather than PDE's are given by [Couder, Douady]. In general, PDE's become relevant if we go beyond average properties of a system and look at geometric properties, those which depend on shape not just size, e.g., the dynamics of coral reef formation [Kaandorp, Sloot].

Is everything a PDE? Is the development of all macroscopic patterns, from galactic formation to animal growth, driven by PDE's? Richard Feynman suggested the answer: "yes" with an eye on the Navier–Stokes equation governing the flow of viscous incompressible fluids, perhaps because the simple elementary principles underlying these equations result in a bewilderingly rich set of patterns such as vortices and Bernard cells.

A biologist knows better than to stick to PDE's and even the physicist's toolkit of nonlinear dynamics used in developmental biology, ecology, and the physiology of the brain, does not appear sufficient to him. Thus the molecular factories of cells are not run by PDE's because, though the underlying biochemistry is local, it is not locally linear. It seems hopeless to model global cellular behavior directly from its underlying biochemistry. Nevertheless, we can focus on specific phenomena: division, growth, differentiation. Imagine a geometric entity composed of billions of cells. One can then ask:

[1] This might look paradoxical at first since chemical reactions and diffusion drive the system down an energy well towards equilibrium, and, moreover, nothing visually interesting occurs in most chemical reactions. Yet, given two or more degrees of freedom, equilibrium (i.e., a fixed point of a vector field) is not necessarily approached along a straight line, but possibly along a spiral in configurations space and such a spiral orbit might easily generate regular oscillating behavior when projected back to space–time, a well known phenomenon in physics. This was discovered in chemistry by Belousov in 1951, a time when every self–respecting chemist would refuse even to consider such nonsense as oscillatory chemistry. Stationary solutions lead for instance to stripes on a tiger and spots on a leopard.

[2] In particular, the PDE's underlying this model (the reaction–diffusion equations) are invariant under translation, and if they are also *linear,* the singularities of their Fourier transform determine the frequencies of observable periodicities, i.e., solutions which exhibit symmetry which is less than the full symmetry of the underlying equation which, in fact, can only admit constants as solutions. The full richness of what we observe is ultimately a result of nonlinearity.

What kind of local division rules generate stable global patterns? Which rules preserve features during growth? Which rules ensure an increase in complexity during morphogenesis? As yet, there is no model for cell division comparable to PDE's in its universality. A mathematically simple model of cellular development is given by *cellular automata.* The idea, due to Ulam, is to fix the spatial structure by placing cells on a regular grid, and then prescribe a local rule by which each cell changes with time: the type of a cell at each generation is determined only by its type and those of its neighbors at the previous moment of time. This may simply seem to be a naïve discrete version of PDE's, but in fact, it produces an immense variety of examples. The most popular example is Conway's game of *Life*, where cells can be thought of as squares on an infinite chessboard and each square touches eight neighbors. The update rule is: a cell which is either *alive* or *dead* is alive only if in the previous generation it had exactly three live neighbors or was alive with two live neighbors. This model is deterministic, i.e., the initial state of a configuration determines its evolution for all time. Though the update rule is simple, anything conceivable can happen given enough time and space! Formally speaking, it was proved by Conway and Gosper that a universal Turing machine can be implemented in the game of Life. Whether random Life configurations eventually lead to universal Turing machines with significant probability is unclear. This question appears to be important in the modeling of biochemical life.

Numerous variations of the game of Life have been proposed to mimic growing colonies of bacteria and natural textures which may be hard to simulate by other mathematical means [Yaroslavsky]. Some cellular automata exhibit configurations which do not change with time. Such stationary solutions are essentially equivalent to *tilings,* partitions of a plane region by translates of a few shapes, e.g., a chessboard tiled by 64 squares. Here again the deceptively simple definition leads to unexpected beauty such as honeycombs and Penrose tilings. One also rapidly encounters complex questions:

Which sets of tiles can tile the plane? Which sets of tiles can only tile the plane aperiodically? It has been shown that for special sets of tiles large regions of the plane may be tiled without there being a tiling of the whole plane, in fact, the problem is undecidable. The point is that a universal Turing machine can be encoded into a suitable tiling and along similar lines one can construct aperiodic tilings [Kari].

Other models of cell growth and division allow the ambient space itself to grow with time. For example, we may take a finite configuration of cells connected to each other according to certain well defined rules and consider a cell division law which preserves the connection conditions. Once again, this model may be inherently complicated, though it has not yet been shown that it can emulate a universal Turing machine [Cannon, Floyd, Parry]. On the other hand, Lindenmayer discovered that there are rules which lead to orderly growth remarkably similar to the development of plants (c.f. [Prusinkiewicz]). For example, consider a root which sprouts two branches each of which simultaneously sprouts its own branches, etc. This results in the standard binary tree which can be made lifelike by randomly curving its branches. Such exponentially growing trees are difficult to fit into Euclidean space (as opposed to hyperbolic space) in which actual plants live. The dichotomy between unlimited growth and the limitation of space leads to mechanical stress in growing plants which somehow regulates the cell division process [Nakielski] [Hejnowicz]. In fact, not all cells can divide indefinitely, only *meristematic* ones, those which did not differentiate in the course of ontogeny [Zagórska–Marek]. Besides, when cells divide, they follow intricate geometric and combinatorial rules [Barlow, Lück, Lück] [Lück, Lück]. This may remind a mathematician of constructions used in the study of three dimensional manifolds.

Though this appears complicated it is simple compared to what happens in a cell. DNA, RNA, proteins, move in harmony tuned by billion years of evolution. We cannot hope to completely model this incredible factory but we can attempt to use some of its machinery. For example, bacteria produce restriction enzymes which cut invading viral DNA at prescribed locations and by using this process, we can split DNA then reassemble it back into new forms and topological shapes: lines, circles, knots, links, graphs [Seeman] [Jonoska] [Head]. The actual process is quite involved, but more and more delicate genetic engineering constructions are continually being developed—a new technological world of biochemical geometry is being created. In principle one can encode everything, once again the universal Turing machine [Paun]. The "biological computer" seems within reach and perhaps ultimately, biological nanotechnology [Seeman].

In our informal investigation of the emergence of patterns we have discussed a number of problems without touching on a fundamental question:

How does the brain perceive shapes? How do human beings perceive elementary components in visual stimuli and what are the laws governing how these components are integrated into a visual whole?

Such questions were addressed at the beginning of this century by the Gestalt school of psychology. It proposed an analysis of human perception in which the main factors favoring perceptual grouping are similarity, contrast, continuation, shapes, proximity, convexity, symmetry, orientation, and parallelism. These concepts might seem rather vague but they turn out to capture a number of important effects both in visual and auditory recognition. The main thrust of the Gestalt school was a list of criteria leading to a partition of visual stimulus into n (or even $\log n$) parts, whereas a naïve division of an image might instead lead to exponentially many components.

Are the above grouping rules still valid when motion is considered? There are indications supporting the fact that dynamic perception seems to influence the sense of spatial continuity, chromatic identity, and uniformity of speeds and trajectories as perceived in static images. It is here that a more subtle study of orientation, perceptual geometry, and stereoscopy come into play. Shapes determined during motion must agree with statically determined shapes and the brain seems to perform a process of *correction* to adapt to both perceptions, as well as a process of *synthesis* leading to the representation of a global shape starting from a small number of clues derived from the visual input [Ninio].

What can we say about the way our brain groups similar objects? For example, how does the brain visually characterize animals versus means of transportation? It seems plausible that there might be a simple mechanism as opposed to the complex linguistic division that we use [Thorpe].

What is the information content of a natural image? How much does an image tell us? From the point of view of Shannon's information theory, a fully randomized picture, where the color and intensity of each pixel in the screen is chosen randomly, carries maximal information so cannot be compressed. Yet in some sense, such an image can be described completely by simply stating that it is "fully randomized." More generally, one can introduce equivalence relations between images which are indistinguishable with respect to a certain class of observations. Examples are the different equivalence classes of pictures indistinguishable to human beings in less than one second and less than five minutes. According to these relations one can isolate essential patterns in images: those seen instantaneously, and those seen after several minutes of staring. The first type of patterns are the most pronounced and are analogous to singularities of smooth maps which correspond in some sense to the patterns classified by the Gestalt school. These local forms in conjunction with spline interpolation and texture generation mechanisms allow up to 1:15–1:50 ratios of compression of natural images. This often surpasses standard techniques such as wavelet decompositions [Briskin, Elichai, Yomdin].

This preface is not meant to be an exhaustive survey of pattern formation but rather an introduction to the topics presented in this collection. We would like to thank all the speakers who participated in the conference "La Formation des Motifs," Bures–sur–Yvette, France, on December 2–6, 1997. We are grateful to all the contributors to this volume for writing papers accessible to a wide audience. Finally, we would like to thank Jean–Michel Morel and Stephen Semmes for their encouragement and stimulating presence, and Helga Dernois, Jacqueline Keller, Marie-Claude Vergne and Maya Schirmann for their help in the production of this volume.

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