Preface

Bifurcation theory is one of the areas in Applied Mathematics that has experienced the most remarkable developments during the last twenty or thirty years. There are at least two reasons for this: first, it has proven to be a very efficient tool for the analysis and computation of solutions of complicated systems, especially partial differential equations such as the Navier-Stokes equations which govern fluid flows or reaction-diffusion equations which appear whenever two or more species (chemical or otherwise) come into contact and interact. The second reason is that it has been realized by the end of the seventies that when a system is invariant under the action of a group of symmetries this can have tremendous consequences on its bifurcations, in particular by producing a rich variety of solutions as well as of unexpected phenomena. It was also realized that most of these phenomena were so strongly associated with the symmetries of the problem that one could use the group-theoretic formalism in order to classify them, and even to compute them. As a by-product, many "strange" behaviors of specific systems such as the well-known Taylor-Couette and Bénard problems in hydrodynamics, came out to be special cases of "universal" properties of systems with the same symmetries. It was therefore completely natural to develop a "new" branch in bifurcation theory, or more correctly to extend this theory so that it accounts for the symmetries. A major contribution in this direction was the book by GOLUBITSKY, STEWART & SCHAEFFER [120], published in 1988, which has been the reference for the past decade in this area. Since the time this book appeared, though, some successes of so-called "Equivariant Bifurcation Theory", such as the description and prediction of fluid flow between rotating cylinders

(Taylor-Couette problem, see CHOSSAT & IOOSS [48]), have led to further developments and applications.

Our aim in this book is primarily to fill this gap between 1988 and present state-of-the-art. Our point of view is deliberately that of Applied Mathematics, which should not mean lack of rigor or "pedestrian" mathematics. It rather means that we shall put some emphasis on infinite-dimensional systems (specifically PDE's of the parabolic type), and that we shall avoid, as much as possible, the geometric constructions of singularity theory and abstract dynamical systems. This does not mean that we consider such approaches as being unimportant or useless; on the contrary we are convinced that the recent contributions of M. Field for example, especially in FIELD [94], have given this theory its most accomplished and powerful formulation. However, this book is difficult to access for the standard applied mathematician (and even for the standard mathematician in general). Moreover it assumes that the reader is somewhat familiar with concepts such as Lie groups, representations, invariant theory, singularity theory, which is not the case for most applied mathematicians. We therefore felt it would be useful to incorporate in a book about equivariant bifurcation theory, those parts of Mathematics which we thought are unavoidable in order to acquire a clear understanding of the objects and phenomena under consideration.

Let us list in a few words what the reader will find in the present book. First, a presentation of the major tools in equivariant bifurcation theory. Some have already been presented elsewhere in the literature. This is the case for the Lyapunov-Schmidt decomposition and for the center manifold and normal form reduction which we recall anyway in Chapters 2 and 3, for the sake of completeness and because it helps to introduce further methods and results. We also tried, as much as possible, not to avoid the analysis of PDE's, restricting ourselves, however, to special cases in which the symmetry allows an elementary treatment of these equations. The reader who would be interested in more general situations, involving sophisticated methods in functional analysis, can consult for example TEMAM [266] and TAYLOR [261, 262, 263]. Other tools, such as invariant theory and orbit space reduction (Chapters 5 and 6), had not yet been presented in a book about equivariant bifurcation and dynamical system.

The main bifurcation theorems are presented in Chapters 7 and 8. In addition to the "classical" equivariant branching lemmas, which are already stated in Chapter 2 as direct applications of the Lyapunov-Schmidt decomposition and elementary group action theory, we describe various results concerning bifurcations with maximal or non-maximal isotropy, and bifurcation of (and from) relative equilibria and relative periodic orbits. Most of these results hold under the assumption that the symmetry group is *compact.* Recently, however, great attention has been given to the case of *non-compact* groups, which occurs in systems (PDE's) posed on an infinitely extended domain. Indeed, it has been recognized that in most physical experiments with spatially extended domains (containers for fluids, vessels for chemical reactions, etc), pattern formation (which is a bifurcation process) led to states which could not be understood from the standard equivariant bifurcation point of view. Let us cite, as an example of special importance, the occurrence of spiral waves in many of these systems when two directions in space are of much larger extension than the third one. The understanding of this pattern formation is beyond the present state of bifurcation theory. However it turns out that the *dynamics* of these "localized" states can still be investigated by the mean of the standard tools of bifurcation theory, after some adaptation due to the non compactness of the symmetry group. Although this activity is still developing at a fast rate, we present one approach in our Chapter 8. The two last chapters are devoted to two important phenomena. One is the bifurcation of "robust" heteroclinic cycles, a kind of flow-invariant object whose existence and stability can lead to spatial and temporal intermittency and the existence of which is strongly dependent on the presence of certain symmetries in the system. The second phenomenon is called *forced symmetry breaking.* This is a natural concept when dealing with systems that arise from physical models. It should be clear that symmetry is an idealization of situations occurring in the physical world. In many cases imperfections of various sorts prevent the system from being completely symmetric. It is then of special interest to analyze how imperfections in the model can affect the solutions. We will see, in particular, that in the case of continuous groups like **O**(3) imperfections can induce very unexpected dynamics, of the same kind as the heteroclinic cycles discussed in the previous chapter. Finally, we provide in the appendices some useful information, mainly about the group **O**(3) and its representations.

It would have been a formidable and somewhat presumptuous task to claim to describe all of the aspects of equivariant bifurcation theory in this book. We have deliberately not treated the following:

Equivariant Hamiltonian systems. These require specific geometrical tools that have been presented elsewhere, see MARSDEN & RATIU [192].

Symmetric attractors and chaos. This is a more "dynamical systems" than "bifurcation theory" oriented topic. The abundant literature which has appeared after the basic paper of CHOSSAT & GOLUBITSKY [45] shows this is a theory of its own, which should deserve a dedicated book or monograph.

Reversible systems and their applications, notably to the study of extended systems in one direction in space. As we shall see in Chapter 7 (Section 8) steady-state or traveling wave systems of elliptic type posed on the real line can be treated as *time reversible* dynamical systems. This powerful idea which was introduced by KIRCHGÄSSNER [163] has led to an impressive development in its application to the study of water waves (more generally of interfacial waves) and other physical problems. There are unavoidable technical difficulties, though, mostly concerning the existence of center manifolds for reversible PDE's, which we did not want to approach in this book. We refer to IOOSS & ADELMEYER [150] for a comprehensive introduction to this idea and to the monograph by MIELKE [201] for a more elaborated theory. See also the book by CHOSSAT & IOOSS [48] for the study of a specific case (the Taylor-Couette problem). Systematic studies of reversible systems can be found in SEVRYUK [242], see also LAMB [176, 177].

The numerical treatment of equivariant bifurcation problems. We decided not to go into this vast subject, because we suffer from insufficient familiarity with it ourselves. Nonetheless, we feel that it is a very important topic which will undergo rapid developments with the fast evolution of computer capabilities. We refer the interested reader to DELLNITZ & WERNER [78], WERNER [277], WERNER & STORK [253], WERNER & SPENCE [278] and GATERMANN [106]. More recent developments can be found in the monography by GOVAERTS [124].

We are conscious of the fact that other lacunae will inevitably show up to the eyes of the specialists reading this book, and we apologize in advance for those. Equivariant bifurcation and dynamical systems are living areas of mathematical research, which continue to evolve and which will certainly offer new and fascinating developments in the future.

To end this introduction, we should like to thank warmly:

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