7

LINEARIZED THEORY OF ELASTICITY

In this chapter we shall discuss the classical theory of elasticity. We shall discuss the general structure of the theory and illustrate the applications of the linearized theory by a few examples.

Rectangular Cartesian coordinates of reference will be used throughout. The coordinates will be denoted by x_1, x_2, x_3 or x, y, z unless stated otherwise.

7.1. BASIC EQUATIONS OF ELASTICITY FOR HOMOGENEOUS ISOTROPIC BODIES

An elastic body has a unique *zero-stress* state, to which the body returns when all stress vanish. All stresses, strains, and particle displacements are measured from this *zero-stress* state: their values are counted as zero in that state.

There are two ways to describe a deformed body: the *material* and the *spatial* (see Sec. 5.2). Consider the spatial description. The motion of a continuum is described by the instantaneous velocity field $v_i(x_1, x_2, x_3, t)$. To describe the strain in the body, a displacement field $u_i(x_1, x_2, x_3, t)$ is specified which describes the displacement of a particle located at x_1, x_2, x_3 at time t from its position in the natural state. Various strain measures can be defined for the displacement field. The Almansi strain tensor is expressed in terms of $u_i(x_1, x_2, x_3, t)$ according to Eq. (4.2:4),

(1) $$e_{ij} = \frac{1}{2}\left[\partial u_j \partial x_i + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right].$$

The particle displacements u_i are functions of time and position. The particle velocity is given by the material derivative of the displacement,

(2) $$v_i = \frac{\partial u_i}{\partial t} + v_j \frac{\partial u_i}{\partial x_j}.$$

The particle acceleration is given by the material derivative of the velocity (5.2:7),

(3) $$\alpha_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

The motion of the body must obey the equation of continuity (5.4:3)

(4) $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0\,,$$

and the equation of motion (5.5:7)

(5) $$\rho \alpha_i = \frac{\partial \sigma_{ij}}{\partial x_j} + X_i\,.$$

In addition to the field Eqs. (4) and (5), the theory of linear elasticity is based on Hooke's law. For a homogeneous isotropic material, this is Eq. (6.2:7).

(6) $$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij}\,,$$

where λ and G are constants independent of the spatial coordinates.†

The famous nonlinear terms in Eqs. (1)–(3) are sources of major difficulty in the theory of elasticity. To make some progress, we are forced to *linearize* by considering small displacements and small velocities, i.e., by restricting ourselves to values of u_i, v_i so small that the nonlinear terms in Eqs. (1)–(3) may be neglected. In such a linearized theory, we have

(7) $$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,,$$

(8) $$v_i = \frac{\partial u_i}{\partial t}\,, \qquad \alpha_i = \frac{\partial v_i}{\partial t}\,.$$

Unless stated otherwise, all that is discussed below is subjected to this restriction of linearization. Fortunately, many useful results can be obtained from this linearized theory.

Equations (1)–(6) or (4)–(8) together are 22 equations for the 22 unknowns ρ, u_i, v_i, σ_i, e_{ij}, σ_{ij}. In the infinitesimal displacement theory we may eliminate σ_{ij} by substituting Eq. (6) into Eq. (5) and using Eq. (7) to obtain the well-known *Navier's equation*,

(9) ▲ $$Gu_{i,jj} + (\lambda + G)u_{j,ji} + X_i = \rho\frac{\partial^2 u_i}{\partial t^2}\,.$$

This can be written in the form

(10) ▲ $$G\nabla^2 u_i + (\lambda + G)e_{,i} + X_i = \rho\frac{\partial^2 u_i}{\partial t^2}\,,$$

†The corresponding equations based on the material description are the following: velocity and acceleration, Eqs. (5.2.3) and (5.2:6); strain measure, the Green's strain tensor, Eq. (4.2:5); the equation of continuity, Eq. (5.2:3); stress tensors, Sec. 16.7; the equations of motion, Eqs. (16.10:1–8); the stress-strain laws, see Sec. 16.11; in particular, Eqs. (16.11:6) and (16.11:7). It can be seen that the kinematical relations appear simpler in the material description, but the equations of motion appear more complicated.

where

(11) $$e = u_{j,j}$$

(12) $$\nabla^2 u_i = u_{i,jj} \,.$$

The quantity e is the *divergence* of the displacement vector u_i. ∇^2 is the *Laplace operator*. If we write x, y, z instead of x_1, x_2, x_3, we have

(13) $$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \,,$$

(14) $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,.$$

Love[1.2] writes Eq. (10) in the form,

(15) $$G\nabla^2(u, v, w) + (\lambda + G)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e + (X, Y, Z)$$
$$= \rho \frac{\partial^2}{\partial t^2}(u, v, w) \,,$$

which is a shorthand for three equations of the type

(16) ▲ $$G\nabla^2 u + (\lambda + G)\frac{\partial e}{\partial x} + X = \rho \frac{\partial^2 u}{\partial t^2} \,.$$

This can also be written as

(17) ▲ $$G\left(\nabla^2 u + \frac{1}{1 - 2\nu}\frac{\partial e}{\partial x}\right) + X = \rho \frac{\partial^2 u}{\partial t^2} \,.$$

If we introduce the rotation vector

(18) $$(\omega_x, \omega_y, \omega_z) \equiv \frac{1}{2}\text{curl}(u, v, w)$$
$$\equiv \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

and use the identity

(19) $$\nabla^2(u, v, w) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e - 2\text{curl}(\omega_x, \omega_y, \omega_z) \,,$$

then Eq. (10) may be written as

(20) $$(\lambda + 2G)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e - 2G\,\text{curl}(\omega_x, \omega_y, \omega_z) + (X, Y, Z)$$
$$= \rho \frac{\partial^2}{\partial t^2}(u, v, w) \,.$$

7.2. EQUILIBRIUM OF AN ELASTIC BODY UNDER ZERO BODY FORCE

Consider the conditions of static equilibrium of an elastic body. If the body force vanishes, $X_i = 0$, then by taking divergence of Eq. (7.1:9), we have

$$Gu_{i,jji} + (\lambda + G)u_{j,jii} = 0\,,$$

or

$$(u_{j,j})_{,ii} = 0\,. \tag{1}$$

In unabridged notations for rectangular Cartesian coordinates, this is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) e = 0\,, \qquad \text{or} \qquad \nabla^2 e = 0\,. \tag{2}$$

Equation (2) is a Laplace equation. A function satisfying Eq. (2) is called a *harmonic function*. Thus, *the dilation e is a harmonic function when the body force vanishes.*

But

$$(3\lambda + 2G)e = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 3\sigma\,,$$

where σ is the mean stress. Hence, *the mean stress is also a harmonic function*:

$$\nabla^2\sigma = 0\,. \tag{3}$$

If we put $X = 0$, $\partial^2 u/\partial t^2 = 0$, and operate on Eq. (7.1:16) with the Laplacian ∇^2, we have

$$(\lambda + G)\frac{\partial}{\partial x}\nabla^2 e + G\nabla^2\nabla^2 u = 0\,.$$

With Eq. (2), this implies that

$$\nabla^4 u = 0\,, \tag{4}$$

where in rectangular Cartesian coordinates,

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + 2\frac{\partial^4}{\partial y^2\partial z^2} + 2\frac{\partial^4}{\partial z^2\partial x^2}\,. \tag{5}$$

Equation (4) is called a *biharmonic* equation, and its solution is called a *biharmonic function*. Hence, *the displacement component u is biharmonic. Similarly, the components v, w are biharmonic.* It follows that *when the* *body force is zero, each of the strain components and each of the stress components, being linear combination of the first derivatives of* u, v, w, *are all biharmonic functions*:

$$\nabla^4 \sigma_{ij} = 0\,, \tag{6}$$

$$\nabla^4 e_{ij} = 0\,. \tag{7}$$

7.3. BOUNDARY VALUE PROBLEMS

Navier's Eq. (7.1:9) combines Hooke's law and the equation of motion. It is to be solved for appropriate boundary and initial conditions. The boundary conditions that occur are usually one of two kinds:

A. *Specified displacements*. The components of displacement u_i are prescribed on the boundary.
B. *Specified surface tractions*. The components of surface traction $\overset{\nu}{T}_i$ are assigned on the boundary.

In most problems of elasticity, the boundary conditions are such that over part of the boundary displacements are specified, whereas over another part the surface tractions are specified. Let the region occupied by an elastic body be denoted by V. Let the boundary surface of V be denoted by S. We separate S into two parts: S_u, where displacements are specified; and S_σ, where surface tractions are specified. Therefore, on S_σ,

$$\overset{\nu}{T}_i = \sigma_{ij}\nu_j = \text{a prescribed function}\,,$$

where ν_j is a unit vector along the outer normal to the surface S_σ. By Hooke's law, this may be written as

$$[\lambda u_{k,k}\delta_{ij} + G(u_{i,j} + u_{j,i})]\nu_j = \text{a prescribed function}\,.$$

Hence, over the entire surface, the boundary conditions are that either u_i, or a combination of the first derivatives of u_i, are prescribed.

In dynamic problems, a set of initial conditions on u_i or σ_{ij} must be specified in the region V and on the boundary S.

The question arises whether a boundary-value problem posed in this way has a solution, and whether the solution is unique or not. The question has two parts. First, do we expect a unique solution on physical grounds? Second, does the specific mathematical problem have a unique solution? In continuum mechanics, there are many occasions in which we do not expect a unique solution to exist. For example, when a thin-walled spherical shell is subjected to a uniform external pressure, the phenomenon of buckling may occur when the pressure exceeds certain specific value. At the buckling load, the shell may assume several different forms of deformation, some stable, some unstable. On the other hand, our everyday experience about the physical world tells us that the vast majority of mechanical cause-and-effect relationships are unique. Theoretically, the physical question is partly answered by thermodynamics. But the mathematical question must be answered by the theory of partial differential equations. A satisfactory theory must bring harmony between the mathematical formulation and the physical world.

In the preceding discussions we have taken the displacements u_i as the basic unknown variables. In problems of static equilibrium, however, it is customary to use an alternate procedure. The equations of equilibrium are first solved for the stresses σ_{ij}. We then use Hooke's law to obtain the strain e_{ij}. This solution will not be unique. In fact, an infinite set of solutions will be found. The correct one is then singled out by the conditions of compatibility. Only the one solution that satisfies both the equations of equilibrium and the equations of compatibility corresponds to a continuous displacement field. This procedure becomes very attractive when stress functions are introduced, which yield general solutions of the equations of equilibrium (see Sec. 9.2).

By means of Hooke's law, the compatibility equation

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \tag{1}$$

can be expressed directly in terms of stress components. On substituting

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\theta\delta_{ij}\,, \qquad \theta = \sigma_{kk}\,,$$

into Eq. (1), we obtain

$$\begin{aligned} \sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} \\ = \frac{\nu}{1+\nu}(\delta_{ij}\theta_{,kl} + \delta_{kl}\theta_{,ij} - \delta_{ik}\theta_{,jl} - \delta_{jl}\theta_{,ik})\,. \end{aligned} \tag{2}$$

Since only six of the 81 equations represented by Eq. (1) are linearly independent, the same must be true for Eq. (2). If we combine Eqs. (2) linearly by setting $k = l$ and summing, we obtain

$$\begin{aligned} \sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} \\ = \frac{\nu}{1+\nu}(\delta_{ij}\theta_{,kk} + \delta_{kk}\theta_{,ij} - \delta_{ik}\theta_{,jk} - \delta_{jk}\theta_{,ik})\,, \end{aligned} \tag{3}$$

which is a set of nine equations of which six are independent because of the symmetry in i and j. Hence, the number of independent equations is not reduced, and Eqs. (2) and (3) are equivalent. Since $\sigma_{kk} = \theta$ and $\sigma_{ij,kk} = \nabla^2\sigma_{ij}$, if we use the equation of equilibrium to replace, say, $\sigma_{ik,kj}$ by $-X_{i,j}$, we can write Eq. (3) as

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\theta_{,ij} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\theta = -(X_{i,j} + X_{j,i})\,, \tag{4}$$

where X_i is the body force per unit volume. In dynamic problems the inertia force should be included in X_i. With a contraction $i = j$, Eq. (4) furnishes a relation between $\nabla^2\theta$ and $X_{i,i}$. If this is used to transform the third term in Eq. (4), we obtain

(5) ▲
$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\theta_{,ij} = -\frac{\nu}{1-\nu}\delta_{ij}X_{k,k} - (X_{i,j} + X_{j.i})\,.$$

Written out *in extenso*, these are

(7)
$$\begin{aligned}
\nabla^2\sigma_{xx} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial x^2} &= -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}\,,\\
\nabla^2\sigma_{yy} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial y^2} &= -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial Y}{\partial y}\,,\\
\nabla^2\sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial z^2} &= -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial Z}{\partial z}\,,\\
\nabla^2\sigma_{yz} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial y\partial z} &= -\left(\frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y}\right),\\
\nabla^2\sigma_{zx} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial z\partial x} &= -\left(\frac{\partial Z}{\partial x} + \frac{\partial X}{\partial z}\right),\\
\nabla^2\sigma_{xy} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial x\partial y} &= -\left(\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}\right).
\end{aligned}$$

These equations were obtained by Michell in 1900, and, for the case in which body forces are absent, by Beltrami in 1892. They are known as the *Beltrami–Michell compatibility equations*.

For a simply connected region, satisfaction of the Beltrami–Michell equations implies that the stress system σ_{ij} is derivable from a continuous displacement field. If the region concerned is multiply connected, additional conditions in the form of certain line integrals must be satisfied (see Sec. 4.7).

7.4. EQUILIBRIUM AND UNIQUENESS OF SOLUTIONS

Consider the problem of determining the state of stress and strain in a body of a given shape which is held strained by body forces X_i and surface tractions $\overset{\nu}{T}_i$. Let us assume that a function $W(e_{11}, e_{12}, \ldots, e_{33})$, called *the strain energy function of the elastic material*, exists and has the property

$$\frac{\partial W}{\partial e_{ij}} = \sigma_{ij} \,. \tag{1}$$

For example, if the material obeys Hooke's law [Eq. (6.1:1)], then

$$W = \frac{1}{2} C_{ijkl} e_{ij} e_{kl} \,. \tag{2}$$

The existence and the positive definiteness of the strain energy function for an elastic body are discussed in Chapter 12. In Sec. 12.4, it is shown that W is positive definite in the neighborhood of the natural state. A natural state of a material is a stable state in which the material can exist by itself in thermodynamic equilibrium. Limiting ourselves to consider materials which have a unique natural state in the theory of elasticity, we are assured of a positive definite strain energy function in the neighborhood of the natural state.

The equations of equilibrium $\sigma_{ij,j} + X_i = 0$ may be written in terms of W as

$$\left(\frac{\partial W}{\partial e_{ij}}\right)_{,j} + X_i = 0 \,. \tag{3}$$

The boundary conditions over the boundary surface $S_u + S_\sigma$ are:

(4a) Over S_u, the values of u_i are given,

(4b) Over S_σ, the tractions $\overset{\nu}{T}_i = \frac{\partial W}{\partial e_{ij}} \nu_j$ are specified

(see Sec. 7.3). If S_σ constitutes the entire surface of the body, it is obvious that equilibrium would be impossible unless the system of body forces and surface tractions satisfy the conditions of static equilibrium for the body as a whole. In the same case, the displacement will be indeterminate to the extent of a possible rigid-body motion.

We shall prove the following theorem due to Kirchhoff. *If either the surface displacements or the surface tractions are given, the solution of the problem of equilibrium of an elastic body as specified by Eqs.* (1)–(4) *is unique in the sense that the state of stress* (*and strain*) *is determinate* *without ambiguity, provided that the magnitude of the stress (and strain) is so small that the strain energy function exists and remains positive definite.*

Proof. Since a strained state must be either unique or nonunique, a proof can be constructed by showing that an assumption of nonuniqueness leads to contradiction. Assume that there exist two sets of displacements u_i' and u_i'' which define two states of strain, both satisfying Eq. (3) and the boundary conditions (4a) and (4b). Then the difference $u_i \equiv u_i' - u_i''$ satisfies the equation

$$\left(\frac{\partial W}{\partial e_{ij}}\right)_{,j} = 0 \tag{5}$$

and the boundary conditions that

$$u_i = 0 \quad \text{on} \quad S_u \quad \text{and} \quad \frac{\partial W}{\partial e_{ij}}\,\nu_j = 0 \quad \text{on} \quad S_\sigma\,. \tag{6}$$

From Eq. (5) we have

$$\int_V u_i \left(\frac{\partial W}{\partial e_{ij}}\right)_{,j} dV = 0\,,$$

which, on integrating by parts, becomes

$$\int_S u_i \frac{\partial W}{\partial e_{ij}}\,\nu_j dS - \int_V \frac{\partial W}{\partial e_{ij}}\,u_{i,j}\,dV = 0\,. \tag{7}$$

The first surface integral vanishes because of the boundary conditions (6). The second volume integral may be written as

$$\int_v \frac{\partial W}{\partial e_{ij}}\,e_{ij}\,dV\,.$$

Now, when W is a homogeneous quadratic function of e_{ij} [Eq. (2)], the integral above is equal to $\int 2W dV$. Since W is assumed to be positive definite, the integral $\int W dV$ cannot vanish unless W vanishes, which in turn implies that $e_{ij} = 0$ everywhere. Hence, $u_i = u_i' - u_i''$ corresponds to the natural, unstrained state of the body. Therefore, the states of strain (and stress) defined by u_i' and u_i'' are the same, contrary to the assumption. Hence, the state of strain (and stress) is unique. Q.E.D.

We must note that the uniqueness theorem is proved only in the neighborhood of the natural state. In fact when the strain energy function fails to remain positive definite, a multi-valued solution or several solutions may be possible.

Problem 7.1. Prove Neumann's theorem that the solution $u_i(x,t)$, for x in $V + S_u + S_\sigma$ and $t \geq 0$, of the following system of equations, is unique.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial e_{ij}}\right) + X_i - \rho\frac{\partial^2 u_i}{\partial t^2} = 0\,, \qquad \text{for } x \text{ in } V,\ t \geq 0\,, \tag{8}$$

$$u_i = f_i(x,t)\,, \qquad \text{for } x \text{ on } S_u,\ t \geq 0\,, \tag{9}$$

$$\frac{\partial W}{\partial e_{ij}}\nu_j = g_i(x,t)\,, \qquad \text{for } x \text{ on } S_\sigma,\ t \geq 0\,, \tag{10}$$

$$u_i = h_i(x)\,, \qquad \dot{u}_i = k_i(x)\,, \qquad \text{when } t = 0, x \text{ in } V + S\,, \tag{11}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,, \tag{12}$$

where X_i, f_i, g_i, h_i, k_i are preassigned functions and $W(e_{ij})$ is a positive definite quadratic form. *Hint*: Multiply Eq. (8) by $\partial u_i/\partial t$ and integrate over V and $(0,t)$. Note that the kinetic energy is positive definite. (Reference, Love, *Elasticity*,[1.2] p. 176.)

Notes on Possible Loss of Uniqueness

The uniqueness theorem of Kirchhoff is the foundation for the method of potentials (Chapters 8 and 9). For, when uniqueness of solution is established, one needs to find only *a* solution of a given boundary-value problem: that solution is *the* solution.

But it is essential for our theory to be able to violate the uniqueness of solution in one way or another. We know that elastic columns can buckle, thin shells may collapse, airplane wings can flutter, machinery can become unstable in one sense or other. The word "stability" has many meanings; to define a stability problem we must define the sense of the word stability. A large class of practical stability problems is connected with the loss of uniqueness of solution. Under certain circumstances two or more solutions may become possible; some of these may be dangerous from engineering point of view or undesirable for the function of the machinery; and the circumstances are said to cause instability.

The uniqueness theorem can be violated by violating any one of its assumptions. Referring to Kirchhoff's theorem, we have two possibilities:

(a) Loss of positive-definiteness of the strain energy function, $W(e_{ij})$.
(b) Basic changes in the equations of equilibrium (or of motion).

The first possibility arises when the material becomes unstable, as yielding or flow occurs (cf. Sec. 12.5). It is relevant to the plastic buckling of columns, plates, and shells.

The second possibility may arise in a variety of forms. The most important are those due to

(1) finite deformation,
(2) nonconservative forces, and
(3) forces that are functionals of the deformation or history of deformation.

Most buckling problems can be understood only if we realize the basic changes in the equation of equilibrium introduced by finite deformation. For finite deformations the equation of equilibrium (or of motion) is given by Eq. (16.10:7) or (16.10:8). These equations are basically nonlinear because the strains and rotations depend on the stresses. The corresponding equations of equilibrium or motion for a plate are given in Sec. 16.16, e.g., Eqs. (16.16:10) to (16.16:14). The situation with a column is similar. The linearized versions of these equations retain the basic features of these large-deflection equations, so that these problems generally become eigenvalue problems or bifurcationa problems.

Nonconservative forces generally depend on the deformation of the structure in a certain specific manner. For example, consider an axial load applied to the end of a cantilever column. If this load is fixed in direction, it is conservative. If this load is not fixed in direction, but may rotate in the process of buckling, then it is nonconservative. The case (3) listed above is a special but broad class of nonconservative forces. It occurs commonly if a solid body is placed in a flowing fluid. The aerodynamic or hydrodynamic pressure acting on the body depends on the deformation of the body, and on the local and whole body velocity and acceleration. If the wake and vorticity are important, the aerodynamic pressure will depend also on the history of deformation. This is commonly the case of an aircraft lifting surface. Under these loadings the terms X_i and $\overset{\nu}{T}_i$ in Eqs. (7.4:3), (7.4:4), and (7.4:8), (7.4:10) are functions or functionals of $u_i(x_1, x_2, x_3, t)$.

In any of the cases mentioned above the basic equations differ from those assumed in the Kirchhoff theorem. Loss of uniqueness does not necessarily follow, but it becomes a possibility and must be investigated.

7.5. SAINT-VENANT'S THEORY OF TORSION

To illustrate the applications of the theory of elasticity, we shall consider the problem of torsion of a cylindrical body. A cylindrical shaft, with an axis z, is acted on at its ends by a distribution of shearing stresses whose resultant force is zero but whose resultant moment is a torque T. The lateral surface of the shaft is stress free. See Fig. 7.5:1.

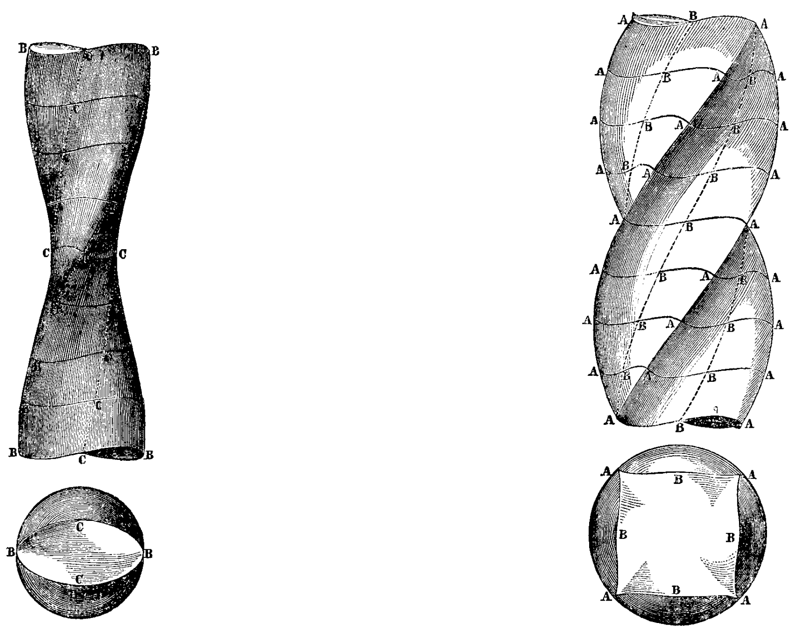

Fig. 7.5:1. Torsion of bars of elliptic and square cross section, as drawn by Saint-Venant.

If the shaft is a circular cylinder, it is very easy to show that all plane cross sections normal to the z-axis remain plane, and the deformation consists of relative rotation θ of the cross sections. The rate of rotation per unit axial length $d\theta/dz$ is proportional to the torque T, with a proportionality constant equal to the product of the shear modulus G of the shaft material, and the polar moment of inertia J of the shaft cross-sectional area:

$$GJ\frac{d\theta}{dz} = T\,. \tag{1}$$

The only nonvanishing component of stress is the shear in cross sections perpendicular to z, whose magnitude is

$$\tau = \frac{Tr}{J}\,, \tag{2}$$

where r is the radius vector from the central axis z.

If the cross section of the shaft is not circular, a plane cross section does not remain plane; it warps, as is shown in Fig. 7.5:1. The problem is to calculate the stress distribution and the deformation of the shaft.

This is an important problem in engineering; for shafts are used to transmit torques and they are seen everywhere. The celebrated solution to the problem is due to Barre de Saint-Venant (1855), who used the so-called *semi-inverse method*, i.e., a method in which one guesses at part of the solution, tries to determine the rest rationally so that all the differential equations and boundary conditions are satisfied. The torsion problem is not simple. Saint-Venant, guided by the solution of the circular shaft, made a brilliant guess and showed that an exact solution to a well-defined problem can be obtained.

We shall consider, then, a cylindrical shaft with an axis z, with the ends at $z = 0$ and $z = L$. A set of rectangular Cartesian coordinates x, y, z will be used, with the x, y-plane perpendicular to the axis of the shaft. The displacement components in the x, y, z-direction will be written as u, v, w, respectively. Saint-Venant assumed that, as the shaft twists, the plane cross sections are warped but the *projections* on the x, y-plane rotate as a rigid body; i.e.,

$$u = -\alpha z y \,, \qquad v = \alpha z x \,, \qquad w = \alpha \varphi(x, y) \,, \tag{3}$$

where $\varphi(x, y)$ is some function of x and y, called the *warping function*, and α is the angle of twist per unit length of the bar and is assumed to be very small ($\ll 1$). We rely on the function $\varphi(x, y)$ to satisfy the differential equations of equilibrium (without body force)

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \,, \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \frac{\partial \sigma_{yz}}{\partial z} &= 0 \,, \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} \frac{\partial \sigma_{zz}}{\partial z} &= 0 \,, \end{aligned} \tag{4}$$

the boundary conditions on the lateral surface of the cylinder

$$\begin{aligned} \sigma_{xx}\nu_x + \sigma_{xy}\nu_y &= 0 \,, \\ \sigma_{yx}\nu_x + \sigma_{yy}\nu_y &= 0 \,, \\ \sigma_{zx}\nu_x + \sigma_{zy}\nu_y &= 0 \,, \end{aligned} \tag{5}$$

and the boundary conditions at the ends $z = 0$ and $z = L$:

$$\sigma_{zz} = 0 \tag{6}$$
$$\sigma_{zx}, \sigma_{zy} \text{ equipollent to a torque } T\,.$$

The constants ν_x, ν_y are the direction cosines of the exterior normal to the lateral surface ($\nu_z = 0$).

From Eq. (3), we obtain the stresses according to Hooke's law.

$$\sigma_{yz} = \alpha G\left(\frac{\partial\varphi}{\partial y} + x\right), \qquad \sigma_{zx} = \alpha G\left(\frac{\partial\varphi}{\partial x} - y\right), \tag{7}$$
$$\sigma_{xy} = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0\,.$$

A substitution of these values into Eqs. (4) shows that the equilibrium equations will be satisfied if $\varphi(x, y)$ satisfies the equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{8}$$

throughout the cross section of the cylinder. To satisfy the boundary conditions (5), we must have

$$\left(\frac{\partial\varphi}{\partial x} - y\right)\cos(\nu, x) + \left(\frac{\partial\varphi}{\partial y} + x\right)\cos(\nu, y) = 0\,, \quad \text{on } C\,, \tag{9}$$

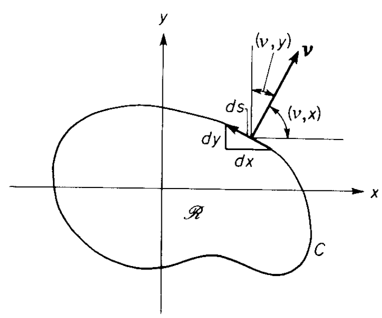

Fig. 7.5:2. Notations.

where C is the boundary of the cross section R (Fig. 7.5:2). But

$$\frac{\partial\varphi}{\partial x}\cos(x, \nu) + \frac{\partial\varphi}{\partial y}\cos(y, \nu) \equiv \frac{\partial\varphi}{\partial\nu}\,;$$

hence, the boundary condition (9) can be written as

$$\frac{\partial\varphi}{\partial\nu} = y\cos(x, \nu) - x\cos(y, \nu) \qquad \text{on } C\,. \tag{10}$$

The boundary conditions (6) are satisfied at the ends $z = 0$ and $z = L$ if

$$\iint_R \sigma_{zx}\, dxdy = 0\,, \qquad \iint_R \sigma_{zy}\, dxdy = 0\,, \tag{11}$$

$$\iint_R (x\sigma_{zy} - y\sigma_{zx})\, dxdy = T\,. \tag{12}$$

We can show that Eqs. (11) are readily satisfied if φ satisfies Eqs. (8) and (10); because

$$\begin{aligned}\iint_R \sigma_{zx}\, dxdy &= \alpha G \iint_R \left(\frac{\partial\varphi}{\partial x} - y\right) dxdy \\ &= \alpha G \iint_R \left\{\frac{\partial}{\partial x}\left[x\left(\frac{\partial\varphi}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\varphi}{\partial y} + x\right)\right]\right\} dxdy\,,\end{aligned}$$

since φ satisfies Eq. (8). On applying Gauss' theorem to the last integral, it becomes a line integral on the boundary C of the region R:

$$\alpha G \int_C x\left[\frac{\partial\varphi}{\partial\nu} - y\cos(x,\nu) + x\cos(y,\nu)\right] ds\,,$$

which vanishes on account of Eq. (10). Similarly, the second of Eqs. (11) is satisfied. Finally, the last condition (12) requires that

$$T = \alpha G \iint_R \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dxdy\,. \tag{13}$$

Writing J for the integral

$$J \equiv \iint_R \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dxdy\,, \tag{14}$$

we have

$$T = \alpha G J\,. \tag{15}$$

This merely shows that the torque T is proportional to the angle of twist per unit length α, with a proportionality constant GJ, which is usually called the *torsional rigidity* of the shaft. The J represents the polar moment of inertia of the section when the cross section is circular and the warping function φ is zero. However, it is conventional to retain the notation GJ for torsional rigidity, even for noncircular cylinders.

Thus, we see that the problem of torsion is reduced to the solution of Eqs. (8) and (10). The solution will yield a stress system σ_{zx}, σ_{zy}. If the end sections of the shaft are free to warp, and if the stresses prescribed on the end sections are exactly the same as those given by the solution, then an exact solution is obtained, and the solution is unique (see Sec. 7.4). If the distribution of stresses acting on the end sections, while equipollent to a torque T, does not agree exactly with that given by Eq. (7), then only an approximate solution is obtained. According to a principle proposed by Saint-Venant, the error in the approximation is significant only in the neighborhood of the end section (see Secs. 10.11–10.13).

Equation (8) is a *potential* equation; its solutions are called *harmonic functions*. The same equation appears in hydrodynamics. The boundary condition (10) is similar to that for the velocity potential in hydrodynamics with prescribed velocity efflux over the boundary. In the hydrodynamics problem, the condition for the existence of a solution φ is that the total flux of fluid across the boundary must vanish. Translated to our problem, the condition for the existence of a solution φ is that the integral of the normal derivative of the function φ, calculated over the entire boundary C, must vanish. This follows from the identity

$$\int_C \frac{\partial \varphi}{\partial \nu} ds = \iint_R \operatorname{div}(\operatorname{grad} \varphi)\, dxdy = \iint_R \nabla^2 \varphi \, dxdy \tag{16}$$

and from the fact that $\nabla^2 \varphi = 0$. This condition is satisfied in our case by Eq. (10), as can be easily shown. Therefore, our problem is reduced to the solution of a potential problem (called Neumann's problem) subjected to the boundary condition Eq. (10).

An alternate approach was proposed by Prandtl, who takes the stress components as the principal unknowns. If we assume that only σ_{xz}, σ_{yz} differ from zero, then all the equations of equilibrium (4) are satisfied if

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0\,. \tag{17}$$

Prandtl observes that this equation is identically satisfied if σ_{xz} and σ_{yz} are derived from a *stress function* $\psi(x, y)$ so that

$$\sigma_{xz} = \frac{\partial \psi}{\partial y}\,, \qquad \sigma_{yz} = -\frac{\partial \psi}{\partial x}\,. \tag{18}$$

This corresponds to the stream function in hydrodynamics, if σ_{xz} and σ_{yz} were identified with velocity components. Although ψ can be arbitrary as far as equilibrium conditions are concerned, the stress system (18) must satisfy the boundary conditions (5) and (6), and the compatibility conditions. From Eq. (7.3:6), we see that compatibility requires that (in the absence of body force),

$$\nabla^2 \sigma_{yz} = 0\,; \qquad \nabla^2 \sigma_{zx} = 0\,,$$

where ∇^2 denotes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) .$$

Hence

(19) $$\frac{\partial}{\partial x} \nabla^2 \psi = 0\,, \qquad \frac{\partial}{\partial y} \nabla^2 \psi = 0\,.$$

It follows that

(20) $$\nabla^2 \psi = \text{const}\,.$$

Of the boundary conditions (5), only the last equation is not identically satisfied. If we note from Fig. 7.5:2, that

(21) $$\nu_x = \cos(\nu, x) = \frac{dy}{ds}\,, \qquad \nu_y = \cos(\nu, y) = -\frac{dx}{ds}\,,$$

we can write the last of Eq. (5) as

(22) $$\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds} = \frac{d\psi}{ds} = 0\,, \qquad \text{on } C\,.$$

Hence ψ must be a constant along the boundary curve C. For a simply connected region, no loss of generality is involved in setting

(23) $$\psi = 0\,, \qquad \text{on } C\,.$$

If the cross section occupies a region R that is multi-connected, additional conditions of compatibility must be imposed (see Sec. 4.7).

It remains to examine the boundary conditions (6). The first, $\sigma_{zz} = 0$, follows the starting assumption. The other conditions are stated in Eqs. (11) and (12). Now,

$$\iint_r \sigma_{zx}\, dxdy = \iint_R \frac{\partial \psi}{\partial y}\, dxdy\,.$$

By Gauss' theorem, this is $\int_C \psi \nu_y ds$, and it vanishes on account of Eq. (23). Similarly, the resultant force in the y-direction vanishes. Thus, Eqs. (11) are satisfied. Finally, Eq. (12) requires that

$$T = -\iint_R \left(x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} \right) dxdy\,,$$

which can be transformed by Gauss' theorem as follows:

$$T = -\iint_R \left\{ \frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi) - 2\psi \right\} dxdy \tag{24}$$

$$= -\int_C \{x\psi \cos(\nu, x) + y\psi \cos(\nu, y)\} ds + \iint_R 2\psi \, dxdy \, .$$

If R is a simply connected region, the line integral vanishes by the boundary condition (23). Hence,

$$T = 2\iint_R \psi \, dxdy \, . \tag{24a}$$

Thus, all differential equations and boundary conditions concerning stresses are satisfied if ψ obeys Eqs. (20), (23), and (24). But there remains an indeterminate constant in Eq. (20). This constant has to be determined by boundary conditions on displacements. We have, from Eqs. (3) and (7),

$$\frac{\partial w}{\partial x} = \frac{\sigma_{zx}}{G} + \alpha y \, , \qquad \frac{\partial w}{\partial y} = \frac{\sigma_{zy}}{G} - \alpha x \, . \tag{25}$$

Differentiating with respect to y and x, respectively, and subtracting, we get

$$\frac{1}{G}\left(\frac{\partial \sigma_{zx}}{\partial y} - \frac{\partial \sigma_{zy}}{\partial x}\right) = -2\alpha \, . \tag{26}$$

Hence, a substitution from Eq. (18) gives

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2G\alpha \, . \tag{27}$$

In this way, the problem of torsion is reduced to the solution of the Poisson Eq. (27) with boundary condition (23).

With either of the two approaches outlined above, the problem of torsion is reduced to standard problems in the theory of potentials in two dimensions. Such potential problems occur also in the theory of hydrodynamics, gravitation, static electricity, steady flow of heat, etc. A great deal is known about these potential problems; many special solutions have been worked out in detail and general methods of solution are available. The most powerful tool for potential theory in two dimensions comes from the theory of functions of a complex variable. Since this branch of applied mathematics is probably well-known to the readers in their study of other branches of physics, we shall not elaborate further. In any case, a detailed account of this beautiful field will require a book in its own name, and indeed many excellent books exist (see, for example, Courant,[5.1] Courant and Hilbert,[10.1]

Kellogg,[5.1] etc.) The complex variable method and the associated singular integral equations approach are developed in the monumental works of Muskhelishvilli;[1.2] a shorter account is given by Sokolnikoff.[1.2] Other methods of solution and detailed examples can be found in the classical books[1.2] of Love, Sokolnikoff, Southwell, Timoshenko and Goodier, Trefftz, Sechler, Green and Zerna. Goodier's article in Flügge's[1.4] *Handbook of Engineering Mechanics* contains results for various cross sections commonly used in engineering.

Example. *Bars with Elliptical Cross Section*

Let the boundary of the cross section (Fig. 7.5:3) be given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\,.$$

Then Eqs. (27) and (23) are satisfied by

$$\psi = -\frac{a^2 b^2 G\alpha}{(a^2+b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right).$$

Equation (24) gives the relation between the torque and the rate of twist $\alpha = d\theta/dz$.

$$T = \frac{\pi a^3 b^3}{(a^2+b^2)}\, G\frac{d\theta}{dz}\,.$$

The stresses are given by Eq. (18). Note that the curves $\psi(x,y) =$ const. have an interesting meaning. The slope dy/dx of the tangent to such a curve is determined by the formula

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0\,.$$

Hence, according to Eq. (18), we have

$$\frac{dy}{dx} = \frac{\sigma_{zy}}{\sigma_{zx}}\,. \tag{28}$$

Thus, at each point of the curve $\psi(x,y) =$ const., the stress vector $(\sigma_{zx}, \sigma_{zy})$ is directed along the tangent to the curve. The curves $\psi(x,y) =$ const. are called the *lines of shearing stress*. The magnitude of the tangential stress is

$$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2} = \sqrt{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2}\,. \tag{29}$$

Hence, τ is equal to the absolute value of the gradient of the surface $z = \psi(x,y)$. The maximum stress occurs where the gradient is the largest.

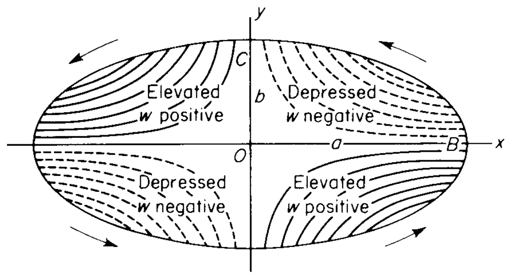

Fig. 7.5:3. Lines of constant warping, according to Saint-Venant.

In the present example, the lines of shearing stress are concentric ellipses. It is easy to see that the spacing of the $\psi = $ const. lines are closest at the end of the minor axis. The maximum shearing stress occurs there and is given by Eq. (29) to be

$$\tau_{\max} = 2G\alpha \frac{a^2 b}{a^2 + b^2}\,.$$

A general theorem can be proved that the points at which the maximum shearing stress occurs lie on the boundary curve of the cross section.

The warping function $\varphi(x, y)$ is easily shown to be

$$\varphi = -\frac{a^2 - b^2}{a^2 + b^2}\, xy\,.$$

Contour lines of constant displacement along the z-axis, $w = \alpha\varphi(x, y) =$ const., are hyperbolas as shown in Figs. 7.5:3, which were taken from Saint-Venant's original publication. The solid lines in the figure indicate where w is positive, the dotted lines where w is negative.

7.6. SOAP FILM ANALOGY

As remarked before, Eqs. (7.5:8) and (7.5:27) occur in many other physical theories entirely unrelated to the torsion problem. For example, if we consider a thin film of liquid, such as that of a soap bubble, we see that the predominant force acting in the film is the surface tension, which may be considered to be constant. The equation of equilibrium of an element of soap film is

$$\frac{T}{R_1} + \frac{T}{R_2} = p\,,$$

where R_1, R_2 are the principal radii of curvature and p is the pressure per unit area normal to the film. The derivation of this equation is simple and can be understood from elementary considerations as illustrated in Fig. 7.6:1. Now if we take a tube whose cross-sectional shape is the same as

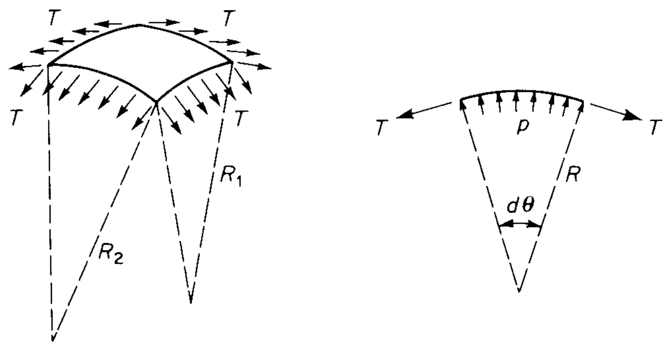

Fig. 7.6:1. Equilibrium of a soap film.

that of the shaft whose torsional property is questioned, cut a plane section, spread a soap film over it under a small pressure p. If the film deflection is sufficiently small, the mean curvature of the film is given by the sum of the second derivatives of the deflection surface. Thus,

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = p\,, \qquad w = 0 \qquad \text{on boundary}\,,$$

where w denotes the deflection of the film measured from the plane of the cross section and x, y are a set of rectangular coordinates. These equations are identical with Eqs. (7.5:23) and (7.5:27). Thus, we obtain Prandtl's *soap film analogy.* The gradient of the soap film is proportional to the shear stress in torsion. The volume under the film and above the cross section is proportional to the total torque.

The value of an analogy lies in its power of suggestion. Most people can visualize the shape of a soap film, perhaps because of their long experience with it. Thus, with the soap film analogy it is very easy to explain the stress concentration at an re-entrant corner in a cross section, such as the points marked by P in Fig. 7.6:2.

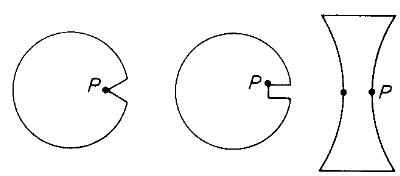

Fig. 7.6:2. Re-entrant corners suggest stress concentration.

Economical and efficient use of materials to transmit forces is an important objective in engineering; and the problem of avoiding the weakest links — points where stress concentrations occur — is obviously of great interest.

7.7. BENDING OF BEAMS

When a shaft is used to transmit bending moments and transverse shear, it is called a *beam*. Since beams are used in every engineering structure, the theory of beams is of great importance. The long history of the development of man's understanding of the action of the beam is a fascinating subject well-recorded in Timoshenko's book.[1.1] Modern investigation began with Galileo, but it is again to the credit of Saint-Venant that the problem is solved within the general theory of elasticity.

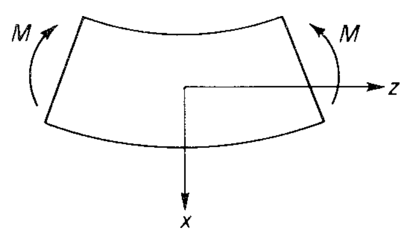

Fig. 7.7:1. Pure bending of a prismatic beam.

Consider first the pure bending of a prismatic beam (Fig. 7.7:1). Let the beam be subjected to two equal and opposite couples M acting in one of its principal planes.† Let the origin of the coordinates be taken at the centroid of a cross section, and let the x, z-plane be the principal plane of bending. The usual elementary theory of bending assumes that the stress components are

$$\text{(1)} \qquad \sigma_{zz} = \frac{Ex}{R}\,, \qquad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0\,,$$

in which R is the radius of curvature of the beam after bending. It is easily verified that the stress system (1) satisfies all the equations of equilibrium (7.5:4) and compatibility (7.3:6). The boundary conditions on the lateral surface of the beam are also satisfied. If the normal stress at the ends of the beam is linearly distributed as in Eq. (1), and if the bending moment is

$$\text{(2)} \qquad M = \iint \sigma_{zz}\, x dx dy = \iint \frac{1}{R} E x^2\, dx dy = \frac{EI}{R}\,,$$

where I is the moment of inertia of the cross section of the beam with respect to the neutral axis parallel to the y-axis, then every condition is satisfied, and Eq. (1) gives an exact solution.

†A principal plane is one that contains the principal axes of the moment of area of the cross sections of the beam.

Consider next the same prismatic beam loaded by a lateral force P at the end $z = L$, and clamped at the end $z = 0$ (Fig. 7.7:2). For a beam so loaded, the resultant shear P has to be resisted by the shearing stresses σ_{zx}, σ_{zy}. Hence, the stress system (1) will not suffice.

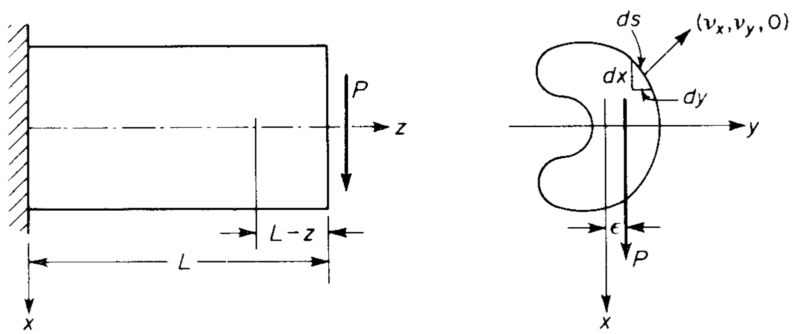

Fig. 7.7:2. Cantilever beam loaded at one end.

Let the force P be parallel to one of the principal axes of the cross section of the beam. (An arbitrary force can be resolved into components parallel to the principal axes, and the action of each component may be considered separately.) Let P be parallel to the x-axis and let the x, z-plane be a principal plane. Saint-Venant, using his semi-inverse method, assumes that

$$\sigma_{zz} = -\frac{P(L-z)x}{I}, \qquad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0, \tag{3}$$

leaving σ_{zx}, σ_{zy} undetermined. The first term is given by the elementary theory, $P(L-z)$ being the bending moment at section z. Now we must see how the equilibrium, compatibility, and boundary conditions can be satisfied. In the absence of body force, the equilibrium equations (7.5:4) require that

$$\frac{\partial \sigma_{zx}}{\partial z} = 0, \qquad \frac{\partial \sigma_{yz}}{\partial z} = 0, \tag{4}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{Px}{I} = 0. \tag{5}$$

The Beltrami–Michell compatibility conditions (7.3:6) require that

$$\nabla^2 \sigma_{yz} = 0, \qquad \nabla^2 \sigma_{xz} + \frac{1}{1+\nu}\frac{P}{I} = 0, \tag{6}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The stress-free boundary condition on the lateral surface requires that [Eq. (7.5:5)]

$$\sigma_{zx} \cos(\nu, x) + \sigma_{zy} \cos(\nu, y) = 0 \,, \qquad \text{on } C \,. \tag{7}$$

The end condition at $z = L$ requires that

$$\sigma_{zz} = 0 \,, \qquad \iint \sigma_{zy} \, dxdy = 0 \,, \tag{8}$$

$$\iint \sigma_{zx} \, dxdy = P \,. \tag{9}$$

The end condition at $z = 0$ is concerned with the conditions of clamping, and it is usually stated in the form

$$u = \frac{\partial u}{\partial z} = 0 \qquad \text{at} \qquad x = y = z = 0 \,. \tag{10}$$

The method of solving Eqs. (4) through (9) is similar to that of Sec. 7.5. Equations (4) imply that both σ_{xz} and σ_{yz} are independent of z. Equation (5) may be written as

$$\frac{\partial}{\partial x}\left(\sigma_{xz} + \frac{Px^2}{2I} - f(y)\right) + \frac{\partial \sigma_{yz}}{\partial y} = 0 \,, \tag{11}$$

where $f(y)$ is a function of y only. Equation (11) can be satisfied identically if the stresses σ_{xz}, σ_{yz} are derived from a stress function $\psi(x, y)$ such that

$$\sigma_{xz} = \frac{\partial \psi}{\partial y} - \frac{Px^2}{2I} + f(y) \,, \qquad \sigma_{yz} = -\frac{\partial \psi}{\partial x} \,. \tag{12}$$

Equations (6) imply that

$$\frac{\partial}{\partial x}(\nabla^2 \psi) = 0 \,, \qquad \frac{\partial}{\partial y}(\nabla^2 \psi) = \frac{\nu}{1+\nu}\frac{P}{I} - \frac{d^2 f}{dy^2} \,. \tag{13}$$

Hence,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy} + c \,. \tag{14}$$

The integration constant c has a very simple physical meaning. Consider the rotation ω of an element of area in the plane of a cross section.

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \,. \tag{15}$$

The rate of change of this rotation in the z-axis direction is

$$
\begin{aligned}
\frac{\partial \omega}{\partial z} &= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y} \qquad (16)\\
&= \frac{1}{2G}\left(\frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial y}\right) = -\frac{1}{2G}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{df}{dy}\right).
\end{aligned}
$$

In deriving the last line, Hooke's law and Eq. (12) are used. Hence, Eq. (14) leads to

$$
-2G\frac{\partial \omega}{\partial z} = \frac{\nu}{1+\nu}\frac{Py}{I} + c\,. \tag{17}
$$

This shows that c represents a constant rate of rotation, i.e., it corresponds to a rigid-body rotation of a cross section (the same kind as in the torsion problem). It can be shown that by a proper shifting of the load P parallel to itself in the plane $z = L$, the torsional deformation can be eliminated so that $c = 0$. (This leads to the concept of *shear center*, the point through which P must act so that $c = 0$.) In the following discussion we shall assume that P acts through the shear center. The more general problem can be solved obviously by a linear superposition of the solutions of bending and of torsion.

On setting $c = 0$, Eq. (14) becomes

$$
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy}\,. \tag{18}
$$

The boundary condition (7) now requires that [see Fig. 7.5:2 and Eqs. (7.5:21)]

$$
\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds} = \frac{\partial \psi}{\partial s} = \left[\frac{Px^2}{2I} - f(y)\right]\frac{dy}{ds}\,. \tag{19}
$$

From these equations, the value of ψ can be determined up to an integration constant which does not contribute anything to the stress system. The function $f(y)$ is arbitrary; it was introduced by Timoshenko to simplify the solution in case the boundary curve of the cross section can be written in the form

$$
C : \frac{Px^2}{2I} - f(y) = 0\,. \tag{20}
$$

This would be the case, for example, if C is a circle or an ellipse. In such a case, we choose $f(y)$ according to Eq. (20). Then the boundary condition may be written as

$$\psi = 0 \qquad \text{on } C\,. \tag{21}$$

It remains to show that the load at the end $z = L$ is equipollent to a shear P, i.e., that Eqs. (8) and (9) are satisfied. This is easily done. For example, using Eqs. (5) and (7), we have

$$\begin{aligned}\iint \sigma_{xz}\,dxdy &= \iint \left[\sigma_{xz} + x\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y}\right) + \frac{Px^2}{I}\right] dxdy \\ &= P + \int_C x[\sigma_{xz}\cos(\nu, x) + \sigma_{yz}\cos(\nu, y)]ds = P\,, \\ \iint \sigma_{yz}\,dxdy &= \iint \left[\sigma_{yz} + y\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y}\right) + \frac{Pxy}{I}\right] dxdy \\ &= \iint y[\sigma_{xz}\cos(\nu, x) + \sigma_{yz}\cos(\nu, y)]ds = 0\,,\end{aligned}$$

since $\iint xydxdy$ vanishes because the x-axis is assumed to be a principal axis.

Thus, all the equations are satisfied if a solution $\psi(x, y)$ is found from Eqs. (18) and (19) or (21). This reduces the beam problem to a standard problem in two-dimensional potential theory.

If a solution is obtained and the moment of the shearing stresses is computed and set equal to $P\epsilon$

$$\iint (x\sigma_{yz} - y\sigma_{xz})\,dxdy = P\epsilon\,, \tag{22}$$

the constant ϵ will give the location of the shear center. The applied load must have an eccentricity ϵ in order to obtain a bending without twisting (see discussion about the constant c above).

Example. *Beam of Circular Cross Section of Radius r*

The boundary curve C is given by the equation

$$x^2 + y^2 = r^2\,.$$

Hence, we take

$$f(y) = \frac{P}{2I}(r^2 - y^2)\,.$$

Equation (18) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1+2\nu}{1+\nu}\frac{Py}{I}\,.$$

It is easily verified that the solution that satisfies the boundary condition (21) is

$$\psi = -\frac{(1+2\nu)}{8(1+\nu)}\frac{P}{I}(x^2+y^2-r^2)y\,.$$

The stress components are

$$\sigma_{xz} = \frac{(3+2\nu)}{8(1+\nu)}\frac{P}{I}\left(r^2 - x^2 - \frac{1-2\nu}{3+2\nu}y^2\right),$$

$$\sigma_{yz} = -\frac{(1+2\nu)}{4(1+\nu)}\frac{Pxy}{I}\,.$$

A large number of examples can be found in the books of Love,[1.1] Sokolnikoff,[1.2] Timoshenko,[1.2] etc. An extensive discussion of the shear center can be found in Sechler's and Sokolnikoff's books.[1.2] There are several possible definitions of shear center and center of twist [Trefftz (1935), Goodier (1944) and Weinstein (1947)]; an outline and comparison of various definitions can be found in Fung's *Aeroelasticity*[10.5] (1957) pp. 471–475. See Bibliography 7.2, p. 881.

7.8. PLANE ELASTIC WAVES

As a further illustration of the theory of elasticity, let us consider some simple types of waves in an elastic medium. The displacement components u_1, u_2, u_3 (or in unabridged notations, u, v, w) will be assumed to be infinitesimal so that all equations are linearized. The basic field equation, in the absence of body force, is Navier's Eq. (7.1:9)

$$\rho\frac{\partial^2 u_i}{\partial t^2} = G u_{i,jj} + (\lambda + G)u_{j,ji}\,. \tag{1}$$

We shall first verify that the motion

$$u = A\sin\frac{2\pi}{l}(x \pm ct)\,, \qquad v = w = 0\,, \tag{2}$$

where A, l, c are constants, is possible if c assumes the special value c_L,

(3) ▲ $$c_L = \sqrt{\frac{\lambda+2G}{\rho}} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}\,.$$

This can be verified at once by substituting Eqs. (2) into (1). The pattern of motion expressed by Eqs. (2) is unchanged when $x \pm c_L t$ remains constant. Hence, if the negative sign were taken, the pattern would move to the right with a velocity c_L as the time t increases. The constant c_L is called the *phase velocity* of the wave motion. In Eqs. (2), l is the *wave length*, as can be seen from the sinusoidal pattern of u as a function of x, at any instant of time. The particle velocity represented by Eqs. (2) is in the same direction as that of the wave propagation (namely, the x-axis). Such a motion is said to constitute a train of *longitudinal waves*. Since at any instant of time the wave crests lie in parallel planes, the motion represented by Eq. (2) is called a train of *plane waves*.

Next, let us consider the motion

$$(4) \qquad u = 0\,, \qquad v = A \sin \frac{2\pi}{l}(x \pm ct)\,, \qquad w = 0\,,$$

which represents a train of plane waves of wave length l propagating in the x-axis direction with a phase velocity c. When Eqs. (4) are substituted into Eqs. (1), it is seen that c must assume the value c_T,

$$(5) \ \blacktriangle \qquad c_T = \sqrt{\frac{G}{\rho}}\,.$$

The particle velocity (in the y-direction) represented by Eqs. (4) is perpendicular to the direction of wave propagation (x-direction). Hence, it is said to be a *transverse wave*. The speeds c_L and c_T are called the characteristic *longitudinal wave speed* and *transverse wave speed*, respectively. They depend on the elastic constants and the density of the material. The ratio c_T/c_L depends on Poisson's ratio only,

$$(6) \ \blacktriangle \qquad c_T = c_L \sqrt{\frac{1-2\nu}{2(1-\nu)}}\,.$$

If $\nu = 0.25$, then $c_L = \sqrt{3}c_T$.

Similar to Eqs. (4), the following example represents a transverse wave in which the particles move in the z-axis direction.

$$(7) \qquad u = 0\,, \qquad v = 0\,, \qquad w = A \sin \frac{2\pi}{l}(x \pm c_T t)\,.$$

The plane parallel to which the particles move [such as the x, y-plane in Eqs. (4), or the x, z-plane in Eqs. (7)], is called the *plane of polarization*.

Table 6.2:1, Sec. 6.2, gives a brief list of the longitudinal wave velocities of some common media. It is very interesting to see that most metals and alloys have approximately the same wave velocities.

Plane waves as described above may exist only in an unbounded elastic continuum. In a finite body, a plane wave will be reflected when it hits a boundary. If there is another elastic medium beyond the boundary, refracted waves occur in the second medium. The features of reflection and refraction are similar to those in acoustics and optics; the main difference is that, in general, an incident longitudinal wave will be reflected and refracted in a combination of longitudinal and transverse waves, and an incident transverse wave will also be reflected in a combination of both types of waves. The details can be worked out by a proper combination of these waves so that the boundary conditions are satisfied. See Sec. 8.14.

7.9. RAYLEIGH SURFACE WAVE

In an elastic body, it is possible to have another type of wave, which is propagated over the surface and which penetrates only a little into the interior of the body. These waves are similar to waves produced on a smooth surface of water when a stone is thrown into it. They are called *surface waves*. The simplest is the *Rayleigh wave* that occurs on the free surface of a homogeneous, isotropic, semi-infinite solid. It is an important type of wave because the largest disturbances caused by an earthquake recorded on a distant seismogram are usually those of Rayleigh waves.

The criterion for surface waves is that the amplitude of the displacement in the medium diminishes exponentially with increasing distance from the boundary.

Let us demonstrate the existence of Rayleigh waves in the simple two-dimensional case. Consider an elastic half-space $y \geq 0$. The surface $y = 0$ is stress-free. Let us consider displacements represented by the real part of the following expressions:

$$
\begin{aligned}
u &= Ae^{-by} \exp[ik(x - ct)] \,, \\
v &= Be^{-by} \exp[ik(x - ct)] \,, \qquad (1) \\
w &= 0 \,,
\end{aligned}
$$

where i is the imaginary unit $\sqrt{-1}$, k is the wave number (a constant), and A and B are complex constants. The coefficient b is supposed to be real and positive so that the amplitude of the waves decreases exponentially with increasing y, and tends to zero as $y \to \infty$.

We would first see if the displacements given by Eq. (1) can satisfy the equations of motion, which, in view of the definitions of c_L and c_T by

Eqs. (7.8:3) and (7.8:5), can be written as

$$\frac{\partial^2 u_i}{\partial t^2} = c_T^2 u_{i,jj} + (c_L^2 - c_T^2) u_{j,ji} \,. \tag{2}$$

Substituting Eqs. (1) into (2), cancelling the common exponential factor and rearranging terms, we obtain the equations

$$\begin{aligned} [c_T^2 b^2 + (c^2 - c_L^2)k^2]A - i(c_L^2 - c_T^2)bkB = 0 \,, \\ -i(c_L^2 - c_T^2)bkA + [c_L^2 b^2 + (c^2 - c_T^2)k^2]B = 0 \,. \end{aligned} \tag{3}$$

The condition for the existence of a nontrivial solution is the vanishing of the determinant of the coefficients, which may be written in the form

$$[c_L^2 b^2 - (c_L^2 - c^2)k^2][c_T^2 b^2 - (c_T^2 - c^2)k^2] = 0 \,. \tag{4}$$

This gives the following roots for b.

$$b' = k\left(1 - \frac{c^2}{c_L^2}\right)^{1/2} , \qquad b'' = k\left(1 - \frac{c^2}{c_T^2}\right)^{1/2} . \tag{5}$$

The assumption that b is real requires that $c < c_T < c_L$. Corresponding to b' and b'', respectively, the ratio B/A can be solved from Eq. (3).

$$\left(\frac{B}{A}\right)' = -\frac{b'}{ik} , \qquad \left(\frac{B}{A}\right)'' = \frac{ik}{b''} . \tag{6}$$

Hence, a general solution of the type (1), satisfying the equations of motion, may be written as

$$\begin{aligned} u &= A' e^{-b'y} \exp[ik(x - ct)] + A'' e^{-b''y} \exp[ik(x - ct)] \,, \\ v &= -\frac{b'}{ik} A' e^{-b'y} \exp[ik(x - ct)] + \frac{ik}{b''} A'' e^{-b'y} \exp[ik(x - ct)] \,, \\ w &= 0 \,. \end{aligned} \tag{7}$$

Now, we would like to show that the constants A', A'', k, and c can be so chosen that the boundary conditions on the free surface $y = 0$ can be satisfied:

$$\sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0 \,, \qquad \text{on } y = 0 \,. \tag{8}$$

By Hooke's law, and in view of Eq. (7), conditions (8) are equivalent to

$$\begin{aligned} &\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0\,, && \text{on } y = 0\,, \\ &\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2G\frac{\partial v}{\partial y} = 0\,, && \text{on } y = 0\,. \end{aligned} \tag{9}$$

On substituting Eqs. (7) into Eq. (9), setting $y = 0$, omitting the common factor $\exp[ik(x - ct)]$, and writing

$$G = \rho c_T^2\,, \qquad \lambda = \rho(c_L^2 - 2c_T^2)\,,$$

we obtain the results

$$\begin{aligned} &-2b'A' - \left(b'' + \frac{k^2}{b''}\right)A'' = 0\,, \\ &\left((c_L^2 - 2c_T^2) - c_L^2\frac{b'^2}{k^2}\right)A' - 2c_T^2 A'' = 0\,. \end{aligned} \tag{10}$$

This can be written more symmetrically, by Eqs. (5), as

$$\begin{aligned} &2b'A' + \left(2 - \frac{c^2}{c_T^2}\right)k^2\frac{A''}{b''} = 0\,, \\ &\left(2 - \frac{c^2}{c_T^2}\right)A' + 2b''\frac{A''}{b''} = 0\,. \end{aligned} \tag{11}$$

For a nontrivial solution, the determinant of the coefficients of A', A'' must vanish, yielding the characteristic equation for c

$$\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\left(1 - \frac{c^2}{c_L^2}\right)^{1/2}\left(1 - \frac{c^2}{c_T^2}\right)^{1/2}. \tag{12}$$

The quantity c^2/c_T^2 can be factored out after rationalization, and Eq. (12), called the *Rayleigh equation*, takes the form

$$\frac{c^2}{c_T^2}\left[\frac{c^6}{c_T^6} - 8\frac{c^4}{c_T^4} + c^2\left(\frac{24}{c_T^2} - \frac{16}{c_L^2}\right) - 16\left(1 - \frac{c_T^2}{c_L^2}\right)\right] = 0\,. \tag{13}$$

If $c = 0$, Eqs. (7) are independent of time, and from Eq. (11) we have $A'' = -A'$ and $u = v = 0$. Hence, this solution is of no interest. The second factor in Eq. (13) is negative for $c = 0$, $c_T < c_L$, and is positive for $c = c_T$. There is always a root c of Eq. (13) in the range $(0, c_T)$. Hence, surface waves can exist with a speed less than c_T.

For an incompressible solid $c_L \to \infty$, Eq. (13) becomes

$$\frac{c^6}{c_T^6} - 8\frac{c^4}{c_T^4} + 24\frac{c^2}{c_T^2} - 16 = 0\,. \tag{14}$$

This cubic equation in c^2 has a real root at $c^2 = 0.91275c_T^2$, corresponding to surface waves with speed $c \simeq 0.95538c_T$. The other two roots for this case are complex and do not represent surface waves.

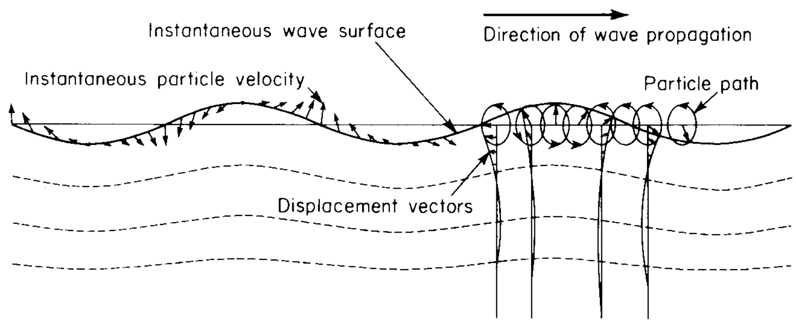

Fig. 7.9:1. Schematic drawing for Rayleigh surface waves.

If the Poisson ratio is $\frac{1}{4}$, so that $\lambda = G$ and $c_L = \sqrt{3}c_T$, Eq. (13) becomes

$$\frac{c^6}{c_T^6} - 8\frac{c^4}{c_T^4} + \frac{56}{3}\frac{c^2}{c_T^2} - \frac{32}{3} = 0\,. \tag{15}$$

This equation has three real roots: $c^2/c_T^2 = 4,\ 2+2/\sqrt{3},\ 2-2/\sqrt{3}$. The last root alone can satisfy the condition that b' and b'' be real for surface waves. The other two roots correspond to complex b' and b'' and do not represent surface waves. In fact, these extraneous roots do not satisfy Eq. (12); they arise from the rationalization process of squaring. The last root corresponds to the velocity

$$c_R = 0.9194c_T\,, \tag{16}$$

which corresponds, in turn, to the displacements

$$\begin{aligned} u &= A'(e^{-0.8475ky} - 0.5773e^{-0.3933ky})\cos k(x - c_R t)\,, \\ v &= A'(-0.8475e^{-0.8475ky} + 1.4679e^{-0.3933ky})\sin k(x - c_R t)\,, \end{aligned} \tag{17}$$

where A' is taken to be a real number, which may depend on k. From Eq. (17), it is seen the particle motion for Rayleigh waves is elliptical retrograde in contrast to the elliptical direct orbit for surface waves on water (see Fig. 7.9:1). The vertical displacement is about 1.5 times the horizontal displacement at the surface. Horizontal motion vanishes at a depth of 0.192 of a wavelength and reverses sign below this.

Figure 7.9:2 shows Knopoff's calculated results for the ratios c_R/c_T, c_R/c_L for Rayleigh waves as functions of Poisson's ratio ν.

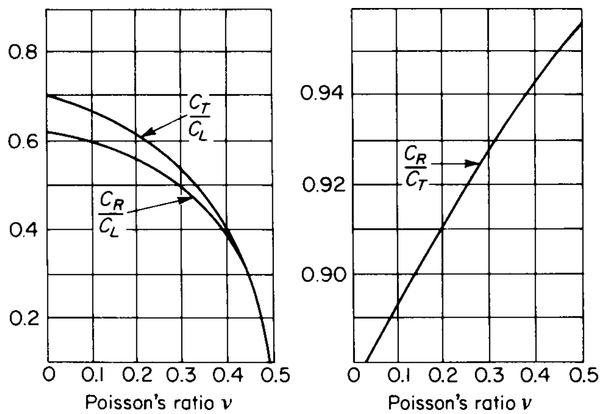

Fig. 7.9:2. Ratios c_R/c_L, c_R/c_T, and c_T/c_L, where c_R = Rayleigh wave speed, c_L = the longitudinal wave speed, c_T = the transverse wave speed.

7.10. LOVE WAVE

In the Rayleigh waves examined in the previous section the material particles move in the plane of propagation. Thus, in Rayleigh waves over the half-space $y \geq 0$ along the surface $y = 0$, propagating in the x-direction, the z-component of displacement w vanishes. It may be shown that surface wave with displacements perpendicular to the direction of propagation (the so-called *SH waves*) is impossible in a homogeneous half-space. However, SH surface waves are observed as

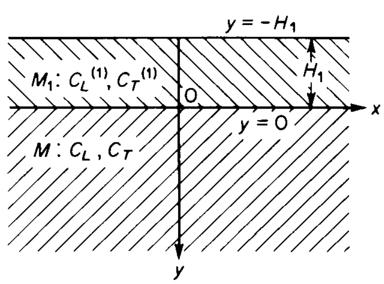

Fig. 7.10:1. A layered half-space.

prominently on the Earth's surface as other surface waves. Love showed that a theory sufficient to include SH surface waves can be constructed by having a homogeneous layer of a medium M_1 of uniform thickness H_1, overlying a homogeneous half-space of another medium M.

Using axes as in Fig. 7.10:1, we take $u = v = 0$, and

$$w = A \exp\left[-k\sqrt{1 - \frac{c^2}{c_T^2}}\,y\right] \exp[ik(x - ct)] \tag{1}$$

in M, and

$$w = \left\{A_1 \exp\left[-k\sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2}\,y\right] + A_1' \exp\left[k\sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2}\,y\right]\right\} \times \exp[ik(x - ct)] \tag{2}$$

in M_1. It is easily verified that these equations satisfy the Navier's equations. If $c < c_T$, then $w \to 0$ as $y \to \infty$, as desired.

The boundary conditions are that w and σ_{xy} must be continuous across the surface $y = 0$, and σ_{zy} zero at $y = -H_1$. On applying these conditions to Eqs. (1) and (2), we obtain

$$A = A_1 + A_1'\,, \tag{3}$$

$$GA[1 - (c/c_T)^2]^{1/2} = G_1(A_1 - A_1')[1 - (c/c_T^{(1)})^2]^{1/2}\,, \tag{4}$$

$$A_1 \exp\{kH_1[1 - (c/c_T^{(1)})^2]^{1/2}\} = A_1' \exp\{-kH_1[1 - (c/c_T^{(1)})^2]^{1/2}\}\,. \tag{5}$$

Eliminating A from Eqs. (3) and (4), and then using Eq. (5) to eliminate A_1 and A_1', we have

$$\frac{G[1 - (c/c_T)^2]^{1/2}}{G_1[1 - (c/c_T^{(1)})^2]^{1/2}} = \frac{A_1 - A_1'}{A_1 + A_1'} = \tanh\{kH_1[1 - (c/c_T^{(1)})^2]^{1/2}\}\,.$$

Hence, we have

$$G\left(1 - \frac{c^2}{c_T^2}\right)^{1/2} - G_1\left[\left(\frac{c}{c_T^{(1)}}\right)^2 - 1\right]^{1/2} \tanh\left\{kH_1\left[\left(\frac{c}{c_T^{(1)}}\right)^2 - 1\right]^{1/2}\right\} = 0 \tag{6}$$

as the equation to give the SH surface wave velocity c in the present conditions.

If $c_T^{(1)} < c_T$, Eq. (6) yields a real value of c which lies in the range $c_T^{(1)} < c < c_T$ and depends on k and H_1 (as well as on G, G_1, c_T, and $c_T^{(1)}$), because for c in this range the values of the left-hand-side terms in Eq. (6) are real and opposite in sign. Thus, SH surface waves can occur under the stated boundary conditions, provided the shear velocity $c_T^{(1)}$ in the upper layer is less than that in the medium M. These waves are called *Love waves.*

Love waves of general shape may be derived by superposing harmonic Love waves of the type (2) with different k. The dependence of the wave speed c on the wave number k introduces a dispersion phenomenon which will be considered later.

PROBLEMS

7.2. Derive Navier's equation in spherical polar coordinates.

7.3. From data given in various handbooks, determine the longitudinal and shear wave speeds in the following materials:

(a) Gases: air at sea level, and at 100,000 ft altitude.
(b) Metals: iron, a carbon steel, a stainless steel, copper, bronze, brass, nickle, aluminium, an aluminium alloy, titanium, titanium carbide, berylium, berylium oxide.
(c) Rocks and soils: a granite, a sandy loam.
(d) Wood: spruce, mahogany, balsa.
(e) Plastics: lucite, a foam rubber.

7.4. Sketch the instantaneous wave surface, particle velocities, and particle paths of a Love wave.

7.5. Investigate plane wave propagations in an anisotropic elastic material. Apply the results to a cubic crystal. *Note*:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_l}{\partial x_j \partial x_k}\,, \qquad u_l = A_l e^{-i(\omega t - k_j x_j)}$$

where $\mathbf{k}(k_1, k_2, k_2)$ is the wave number vector normal to the wave front.

7.6. Determine the stress field in a rotating, gravitating sphere of uniform density.