Lars Valerian Ahlfors
Curriculum Vitae

Born

April 18, 1907

Degrees

Fil.-Kand.	University of Helsinki	1928
Fil.-Lic.	University of Helsinki	1930
Fil. Mag. - Fil. Dr.	University of Helsinki	1932
M.A. (complimentary)	Harvard University	1945
Ll.D. (h.c.)	Boston University	1953
Fil.-Dr. (h.c.)	Abo Adademi	1970
Ph.Dr. (Fakultat II, h.c.)	University of Zürich	1977
Sc.D. (h.c.)	University of London	1978

Positions Held

Abo Adademi	Lecturer	1929–1933
University of Helsinki	Adjunct	1933–1935
Harvard University	Lecturer	1935–1938
University of Helsinki	Professor	1938–1944
University of Zürich	E.O. Professor	1945–1946
Harvard University	Professor	1946–1977
Harvard University	Emeritus Professor	1977–

Visiting Appointments

Mittag-Leffler Institute	1972
Columbia University	Spring 1978
University of Michigan	Fall 1979

Prizes

Fields' Medal	1936
International Prize (Finland)	1968
Wolf Prize in Mathematics	1981

Memberships

Societas Scientiarum Fennica
Academia Scientiarum Fennica
National Academy of Sciences
Swedish Royal Academy of Science
Danish Royal Academy of Science

PUBLICATIONS OF LARS V. AHLFORS

1929 [1] Sur le nombre des valeurs asymptotiques d'une fonction entière d'ordre fini. *C. R. Acad. Sci. Paris 188: 688-689.*

[2] Über die asymptotischen Werte der ganzen Funktionen endlicher Ordnung. *Ann. Acad. Sci. Fenn.*, Ser. A., 32, 6: 1-15

1930 [3] Sur quelques propriétés des fonctions méromorphes. *C. R. Acad. Sci. Paris* 189: 720-722.

[4] Beiträge zur Theorie der meromorphen Funktionen. *7th Scand. Math. Congr.*, Oslo, pp. 84-87.

[5] Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen. *Acta Soc. Sci. Fenn.*, N. S. A1, pp. 1-40.

1931 [6] Zur Bestimmung des Typus einer Riemannschen Fläche. *Comm. Math. Helv.* 3: 173-177.

[7] Ein Satz von Henri Cartan und seine Anwendung auf die Theorie der meromorphen Funktionen. *Comm. Phys. Math. Soc. Sci. Fenn.* 5, 16: 1-19.

1932 [8] Sur une généralisation du théorème de Picard. *C. R. Acad. Sci. Paris* 194: 245-247.

[9] Sur les fonctions inverses des fonctions méromorphes. *C. R. Acad. Sci.* 194: 1145-1147.

[10] Über die asymptotischen Werte der meromorphen Funktionen endlicher Ordnung. *Acta Acad. Aboensis* 6: 1-8.

[11] Über eine in der neueren Wertverteilungstheorie betrachtete Klasse transzendenter Funktionen. *Acta Mathematica*, pp. 375-406.

[12] Ein Satz über die charakteristische Funktion und den Maximalmodul einer meromorphen Funktion. *Soc. Sci. Fenn. Comm. Math. Phys.* 6, no. 9: 1-4.

[13] Quelques propriétés des surfaces de Riemann correspondant aux fonctions méromorphes. *Bull. Soc. Math. France* 60: 197-207.

1933 [14] Über, die Kreise die von einer Riemannschen Fläche schlicht überdeckt werden. *Comm. Math. Helv.* 5: 28-38.

[15] Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée. *Acta Soc. Sci. Fenn.*, N. S. 2: 1-17.

1935 [16] Über eine Methode in der Theorie der meromorphen Funktionen. *Soc. Sci. Fenn. Comm. Phys.-Math.* 8, 10: 1-14.

[17] Sur le type d'une surface de Riemann. *C. R. Acad. Sci. Paris* 201: 30-32.

[18] Über die konforme Abbildung von Überlagerungsflächen. *8th Scand. Congr. of Math.* (Stockholm, 1934), pp. 299-305.

[19] Zur Theorie der Überlagerungsflächen. *Acta Math.* 65: 157-194.

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1936 [20] Über eine Klasse von Riemannschen Flächen. *Soc. Sci. Fenn. Comm. Phys.-Math.* 9: 6.

1937 [21] On Phragmén-Lindelöf's Principle. *Trans. Amer. Math. Soc.* 41, no. 1: 1-8.

[22] Über die Anwendung differentialgeometrischer Methoden zur Untersuchung von Überlagerungsflächen. *Acta Soc. Sci. Fenn.*, N. S. A, Tom II (6) pp. 1-17.

[23] (With H. Grunsky) Über die Blochsche Konstante. *Math Zeitschr.* 42: 671-673.

[24] Geometrie der Riemannschen Flächen. *Comptes rendus du congrès international des mathématiciens*, Oslo (1936), pp. 239-248.

1938 [25] An Extension of Schwarz's Lemma. *Trans. Amer. Math. Soc.* 43, no. 3: 359-364.

1939 [26] Zur Uniformisierungstheorie. *Comptes rendus du congrès des mathématiciens scandinaves à Helsinki* (1938), pp. 235-248.

1941 [27] The Theory of Meromorphic Curves. *Acta Soc. Sci. Fenn.* N. S. A, Tom III (4) pp. 1-31.

1943 [28] Die Begründung des Dirichletschen Prinzips. *Soc. Sci. Fenn. Comm. Phys.-Math.* 11, 15: 1-15.

1946 [29] (With A. Beurling) Invariants conformes et problèmes extrémaux. *Comptes rendus du 10^{me} congrès des mathématiciens scandinaves*, Copenhagen, pp. 341-351.

1947 [30] Bounded analytic functions. *Duke Mathematical Journal* 14, 1: 1-11.

[31] Normalintegrale auf offenen Riemannschen Flächen. *Annales Acad. Scient. Fenn.*, Ser. A1, no. 35.

[32] Das Dirichletsche Prinzip. *Math. Annalen* 120: 36-42.

1948 [33] Book review: Topological methods in the theory of functions of a complex variable by Marston Morse. *Bull. Amer. Math. Soc.* 54, no. 5: 489-491.

1949 [34] Questions of regularity connected with the Phragmén-Lindelöf principle. *Annals of Math.* 50, no. 2: 341-346.

1950 [35] (With A. Beurling) Conformal invariants and function-theoretic null-sets. *Acta Mathematica* 83: 100-129.

[36] Open Riemann surfaces and extremal problems on compact subregions. *Commentarii Mathematici Helvetici* 24: 100-129.

1951 [37] Book review: Coefficient regions for schlicht functions by A. C. Schaeffer and D. C. Spencer. *Bull. Amer. Math. Soc.* 57: 328-331.

[38] Remarks on the classification of open Riemann surfaces. *Ann. Acad. Sci. Fenn.*, Ser. A1, 87 pp. 1-8.

1952 [39] Remarks on the Neumann-Poincaré integral equation. *Pacific J. Math.* 2: 271-280.

[40] On the characterization of hyperbolic Riemann surfaces. *Ann. Acad. Sci. Fenn.*, Ser. A1, Math.-Phys. 125 p. 1925.

[41] (With A. Beurling) Conformal invariants. Construction and applications of con-

formal maps. Proceedings of a symposium, pp. 243-245. National Bureau of Standards, Appl. Math. Ser. no. 18, U.S. Government Printing Office.

[42] (With H. L. Royden) A counterexample in the classification of open Riemann surfaces. *Ann. Acad. Sci. Fenn.*, Ser. A1, Math-Phys. 120: (5).

1953 [43] Development of the theory of conformal mapping and Riemann surfaces through a century. Contributions to the theory of Riemann surfaces. *Annals of Math. Studies* 30: 3-13. Princeton, N.J.: Princeton University Press.

1954 [44] On quasiconformal mappings. *Jour. d'Analyse Math* 3: 1-58.

1955 [45] Conformality with respect to Riemannian metrics. *Ann. Acad. Sci. Fenn.*, Ser. A1, no. 206: 22.

[46] Two numerical methods in conformal mapping. Experiments in the computation of conformal maps, pp. 45-52. National Bureau of Standards Appl. Math. Series, U.S. Government Printing Office.

[47] Remarks on Riemann surfaces. *Lectures on functions of a complex variable*, pp. 45-48. Ann Arbor; University of Michigan Press.

1956 [48] Square integrable differentials on open Riemann surfaces. *Proc. Nat. Acad. Sci.* 42: 758-760.

[49] (With A. Beurling) The boundary correspondence under quasiconformal mappings. *Acta Math.* 96: 125-142.

1958 [50] Extremalprobleme in der Funktionentheorie. *Ann. Acad. Sci. Fenn.*, Ser. A1, no. 249/1: 9.

[51] The method of orthogonal decomposition for differentials on open Riemann surfaces. *Ann. Acad. Sci. Fenn.*, Ser. A1, no. 249/7: 15.

[52] Abel's theorem for open Riemann surfaces. *Seminars on Analytic Functions*, vol. II, pp. 7-19. Institute for Advanced Studies, 1957.

1960 [53] (With L. Bers) Riemann's mapping theorem for variable metrics. *Annals of Math.* 72 pp. 385-404.

[54] The complex analytic structure of the space of closed Riemann surfaces. *Analytic Functions*, pp. 45-66. Princeton, N.J.: Princeton University Press.

1961 [55] Classical and contemporary analysis. *SIAM Reviews.* 3: 1-9.

[56] (With L. Bers) Spaces of Riemann surfaces and quasiconformal mappings. *Iz dat. Inostr. Lit. Moscow*, 177.

[57] Some remarks on Teichmüller's space of Riemann surfaces. *Ann. of Math.* 74: 171-191.

1962 [58] Curvature properties of Teichmüller's space. *Jour. d'Analyse Math.* 9: 161-176.

[59] Geodesic curvature and area. *Studies in mathematical analysis and related topics.* Stanford, Calif.: Stanford University Press.

[60] (With G. Weill) A uniqueness theorem for Beltrami equations. *Proc. Amer. Math. Soc.* 13: 975-978.

1963 [61] Teichmüller spaces. *Proc. International Congr. Math.* (1962) pp. 3-9. Inst. Mittag-Leffler.

[62] The use of quasiconformal deformations for the study of Kleinian groups. Outlines Joint Symp. Partial Differential Equations (Novosibirsk 1963), pp. 10-13. Acad. Sci., USSR, Siberian Branch, Moscow.

[63] Quasiconformal reflections. *Acta Math.* 109: 291-301.

1964 [64] Extension of quasiconformal maps from two to three dimensions. *Proc. Nat. Acad. Sci.* 51: 768-771.

[65] Finitely generated Kleinian groups. *Am. J. Math.* 86 pp. 413-429 and 87 p. 759.

[66] Eine Bemerkung über Fuchssche Gruppen. *Math. Z.* 84: 244-245.

[67] Kleinian groups. *Scripta Math.* 27: 97-103.

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1965 [69] The modular function and geometric properties of quasi-conformal maps. *Proc. Conf. Complex Analysis* (Minneapolis, 1964), pp. 296-300. Berlin: Springer.

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1966 [71] Fundamental polyhedrons and limit point sets of Kleinian groups. *Proc. Nat. Acad. Sci.* 55: 183-187.

[72] Remarks on Carleman's formula for functions in a half-plane. *SIAM J. Numer. Anal.* 3: 183-187.

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1967 [74] Hyperbolic motions. *Nagoya Math. J.* 29: 163-165.

1968 [75] Eichler integrals and the area theorem of Bers. *Michigan Math. J.* 15: 257-263.

1969 [76] The structure of a finitely generated Kleinian group. *Acta Math.* 122: 1-17.

1971 [77] Two lectures on Kleinian groups. Proc. Romanian-Finnish Seminar on Teichmüller spaces and quasiconformal mappings, Brasov, Romania, 1969.

[78] Remarks on the limit point set of a finitely generated Kleinian group. *Advances in the theory of Riemann surfaces* (Stony Brook 1969). Princeton, N.J.: Princeton University Press.

1973 [79] Sufficient conditions for quasiconformal extension. Proc. conf. University of Maryland.

1974 [80] Conditions for quasiconformal deformations in several variables. *Contributions to analysis.* New York: Academic Press.

[81] A remark on schlicht functions with quasiconformal extensions. Proc. Symp. Complex Analysis, Kent University.

1975 [82] Invariant operators and integral representations in hyperbolic space. *Math. Scand.* 36: 27–43.

1976 [83] Quasiconformal deformations and mappings in R^n. *Jour. d'Analyse Math.* 30: 74–97.

[84] Das mathematische Schaffen Rolf Nevanlinnas. *Ann. Acad. Sci. Fenn.*, Ser. A1, 2: 1-15.

[85] On a class of quasiconformal mappings. *Österreich. Akad. Wiss. Math.-Naturw.*, Kl. S. BII 185, no. 1-3: 5-10.

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1977 [87] A somewhat new approach to quasiconformal mappings in R^n. Complex Analysis, Kentucky, *Lecture Notes in Mathematics*, New York: Springer Verlag. pp. 1–6.

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1978 [89] A singular integral equation connected with quasiconformal mappings in space. *L'Enseignement Mathématique*, II^{me} Série, Fascicule 3–4.

[90] Quasiconformal mappings, Teichmüller spaces, and Kleinian groups. Proc. of the international congress of mathematicians, Helsinki, 1978.

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[92] Ergodic properties of groups of Möbius transformations. Proc. of a conference in Kozubnik, Poland, 1979. *Lecture Notes in Mathematics* 798: 1-8.

1982 [92] Rolf Nevanlinna och riemannska vtor. Arkhimedes, Rolf Nevanlinna-erikoisnumero, pp. 30–33.

[93] Riemann surfaces and small point sets. *Ann. Ac. Sci. Fenn., Series A. I. Math.* 7: 49–57.

1984 [94] Old and new in Möbius groups. *Ann. Ac. Sci. Fenn. Series, A. I. Math.* 9: 93–105.

1985 [95] Möbius transformations and Clifford numbers. Dedicated to the memory of Harry Ernest Rauch. Berlin, Heidelberg, Springer-Verlag.

1986 [96] On the fixed points of Möbius transformations in R^n. *Ann. Ac. Sci. Fenn. Series A. I. Math.* 10: 15–27.

[97] Möbius transformations in R^n expressed through 2×2 matrices of Clifford numbers. *Complex Variables* 5: 215–224.

1988 [98] Cross-ratios and Schwarzian derivatives in R^n. *Complex Analysis*, Vol. Dedicated to Professor Albert Pfluger. Basel, Birkhäuser Verlag.

1989 [99] (with Pertti Loanesto), Some remarks on Clifford algebras. *Complex Variables* 12: 201–209.

1991 [100] The binomial theorem in the algebra A^r. In Constantin Carathéodorg: An International Tribute. World Scientific.

Books

[101] *Complex Analysis*, New York: McGraw-Hill. 1st ed. (1953), 2nd ed. (1966), 3rd ed. (1979).

[102] *Riemann Surfaces* (with L. Sario), Princeton, N.J.: Princeton University Press (1960).

[103] *Lectures on Quasiconformal Mappings*, New York: D. Van Nostrand Company (1966).

[104] *Conformal Invariants*, New York: McGraw-Hill (1973).

Ahlfors' Preface to his Collected Papers

When first confronted with the prospect of having my collected papers published, I felt both awe and confusion, but I calmed down when I realized that the purpose was not to honor the author, but to be of service to the mathematical community. If young scholars of a future generation should desire to find out what some mathematicians of the twentieth century were up to, they would indeed have reason to be thankful if spared the need to seek this information from a multitude of sources.

As an introduction it seems polite and useful to begin with a brief outline of my life, especially as related to my professional activity. I was born the eighteenth of April 1907 in Helsingfors, Finland. My father was a professor of mechanical engineering at the Polytechnical Institute. My mother died in childbirth when I was born.

At the time of my early childhood Finland was under Russian sovereignty, but with a certain degree of autonomy, sometimes observed and sometimes disregarded by the czar who was, by today's standards, a relatively benevolent despot. Civil servants, including professors, were able to enjoy a fairly high standard of living, a condition that was to change radically during World War I and the Russian revolution that followed.

As a child I was fascinated by mathematics without understanding what it was about, but I was by no means a child prodigy. As a matter of fact I had no access to any mathematical literature except in the highest grades. Having seen many prodigies spoiled by ambitious parents, I can only be thankful to my father for his restraints. The high school curriculum did not include any calculus, but I finally managed to learn some on my own, thanks to clandestine visits to my father's engineering library.

I entered Helsingfors University in 1924 and soon realized how very fortunate I was to have two truly outstanding mathematicians as teachers, Ernst Lindelöf and Rolf Nevanlinna. At that time the university was still run on the system of one professor for each subject. Fortunately, the need for more professors had become acute and Nevanlinna soon occupied the second chair in mathematics. The elementary teaching was in the hands of two "adjunct professors" and several part-time teachers with the title of "docent." There were essentially no courses on the graduate level; advanced reading was done under the supervision of Lindelöf.

Ernst Lindelöf is rightly considered the father of mathematics in Finland. In the 1920s *all* Finnish mathematicians were his students. He was essentially self-taught and found much of his inspiration in the works of Cauchy. His worldwide reputation as a leading complex analyst was well founded, but when I knew him he had given up research in favor of teaching, at which he was a master. I still remember many Saturday mornings when I had to visit him in his home at 8 A.M. to be praised or scolded – as the case may have been.

In the spring of 1928 I earned my degree of "fil. kand," a title that was to be changed to the equivalent "fil. mag." at a public "promotion." In the fall term of 1928 Hermann Weyl was on leave from ETH in Zürich and Rolf Nevanlinna had been invited to take his place. At the urging of Lindelöf, my father agreed to let me go along to Zürich, no doubt at a nontrivial sacrifice. He may have been moved by the fact that he himself spent some time at ETH as a young man. This was my first trip abroad, and I found myself suddenly transported from the periphery to the center of Europe.

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It was also my first exposure to live mathematics. Nevanlinna was a young man of 33 who had already won widespread acclaim, and I was a very immature 21. The course covered contemporary function theory, including the main parts of Nevanlinna's theory of meromorphic functions, and was essentially a forerunner of his famous "Eindeutige analytische Funktionen." Among other things, Nevanlinna introduced the class to Denjoy's conjecture on the number of asymptotic values of an entire function, including Carleman's partial proof. I had the incredible luck of hitting upon a new approach, based on conformal mapping, which with very considerable help from Nevanlinna and Pólya led to a proof of the full conjecture. With unparalleled generosity they forbade me to mention the part they had played, and Pólya, who rightly did not trust my French, wrote the Comptes Rendus note. For my part I have tried to repay my debt by never accepting to appear as coauthor with a student.

With a small grant from a student organization I was able to follow Nevanlinna to Paris for three more months. There I discovered a geometric interpretation of the Nevanlinna characteristic, which, as it turned out, had been found independently by Shimizu in Japan. Nevertheless, this was the upbeat to an intense involvement with meromorphic functions.

On my return to Finland I entered my first teaching assignment as lecturer (lektor) at Åbo Akademi, the Swedish-language university in Åbo (Turku). At the same time I began work on my thesis, which I defended in the spring of 1930. For formal reasons my degree of Ph.D. was delayed until 1932.

During 1930-1932 I made several trips to continental Europe, including a longer stay in Paris, with a grant from the Rockefeller Institute. My name was becoming known, and I met many of the leading mathematicians. In 1933 I was able to return to Helsingfors as adjunct professor. That same year I married Erna Lehnert, a girl from Vienna, who with her parents had settled first in Sweden and then in Finland. This was the happiest and most important event in my life.

Our life in Helsingfors was pleasant, but uneventful. Quite unexpectedly I received a letter from Harvard University, which to me was hardly more than a name. It turned out that the Mathematics Department was looking for a young mathematician, and Carathéodory, whom I had met in Munich, had recommended me warmly. I was hesitant at first, but after persuasion by my sponsor W. C. Graustein I agreed to a trial period of three years beginning in the fall of 1935. We found life in Cambridge very rewarding, although the social life seemed somewhat Victorian. However, the friendliness of my colleagues was absolutely disarming. I also found that my knowledge of mathematics was still rather spotty, and I learned a lot during these three years.

I was in for the surprise of my life when in 1936, at the International Congress in Oslo, I was told only hours before the ceremony that I was to receive one of the first two Fields medals ever awarded. The prestige was perhaps not yet the same as it is now, but in any case I felt singled out and greatly honored. The citation by Carathéodory mentions explicitly my paper "Zur Theorie der Überlagerungsflächen," which threw some new light on Nevanlinna's theory of meromorphic functions. The award contributed in great measure to the confidence I felt in my work.

In the spring of 1938 I had to decide whether to stay at Harvard or return to Finland, where I was offered a professorship at the University of Helsinki. Lindelöf, who was already retired, urged me to return home as a "patriotic duty." In the end I think it was plain homesickness that decided my return, and back in Finland we enjoyed a very happy year together with our firstborn child.

Alas, the happiness did not last. The war broke out, and it became clear that Finland would not be spared. My wife and children – the second a newborn infant – were evacuated and found refuge with relatives in Sweden. The university was closed for lack of male students, but otherwise life went on, partly in air-raid shelters. Because of an earlier physical condition, I had never been called to military duty, and my only part in the war effort was as an insignificant link in a communications setup.

Soon after the end of the Winter War, my family was able to return home and resume a seemingly normal life. Politics in Finland took an unfortunate turn, however, and when Hitler attacked the Soviet Union in 1941 Finland was his ally. When the Russians were finally able to repulse the attack, they could also intensify the war in Finland with foreseeable results. The Finnish-Russian war ended with a separate armistice in September 1944, whereupon Finland was able to expulse the German troops stationed there. The harsh terms of the armistice left Finland in a very difficult position.

Although the uncertainty of the future and the suffering of the bereaved were on everybody's mind, the wartime was not a complete loss. It unified the nation and paved the way for a return to relatively stable conditions. Paradoxically, I was myself able to do a lot of work during the war, although without the benefit of accessible libraries.

During the summer of 1944, I received an offer from the University of Zürich. Because I saw this as my only opportunity to be reunited with my family and in view of the bleak future in Finland, I accepted in principle although for the moment I saw no physical possibility of following through the invitation. My health had declined, and, because I had no military duties, I was allowed to go to Sweden to recuperate. The problem was now to get from Sweden to Switzerland. By that time Finland was at war with Germany and, at least on paper, also with England. An appeal to the German legation in Stockholm proved fruitless, but the British were willing to let us pass through Great Britain if an opportunity arose. The Swedes had organized some semiregular "stratospheric flights" on moonless nights from Stockholm to Prestwick, Scotland. With the help of diplomatic channels we managed to be placed on the list of potential passengers. Obtaining permission from the British was still necessary and depended on the military situation. Unfortunately, when our low priority had made us eligible, the Battle of the Bulge put a temporary stop to our hopes. But, finally, one day in March 1945 we were told to be ready to leave, weather permitting. It is difficult to forget that flight. The plane was a reconditioned Flying Fortress, with perhaps a dozen passengers. It was not pressurized, and breathing was accomplished by individual oxygen masks. Swim vests were worn by all. Our children, ages 5 and 6, were quite capable of understanding the implications of danger.

We left Sweden with feelings of deep gratitude. Virtually penniless, we had been taken care of not only by close relatives, but also by mathematical colleagues, who made it possible to stay in Uppsala for months. I am forever indebted to Arne Beurling, who showed what true friendship can be.

An arduous train ride took us from Glasgow to London, where we had to wait several days for the Channel to be cleared for a ferry to Dieppe. Much of the time was spent in the London zoo, but the frequent explosions of V2 rockets made us wonder if this was wise.

To cut a long story short, we were finally shipped to Paris, where the Swiss legation was unforgettably helpful in securing lodgings in a luxury hotel – and in providing Swiss cigars to the station master. On arriving at the Swiss border, somewhat disheveled and again penniless, we were met by the Red Cross, who lent us Swiss money. Although there was some rationing even in Switzerland, we were overwhelmed by the chocolates, cakes, and other foods, the likes of which we had not seen for years.

The University of Zürich was ready to begin the summer term. I was met by the Director of the

Mathematics Institute, Professor R. Fueter, in the single room that served as office for the institute and all the professors. My first disappointment came when I learned that I would be responsible for Descriptive Geometry, a subject that for some reason had survived in the Swiss high school and undergraduate curriculum. My second shock was that it was to be taught from 7 to 9 o'clock in the morning.

Nevertheless, I slowly adjusted to my work, which even included some serious, although not very advanced, mathematics. Professor Fueter and his colleague Professor Finsler were getting on in years, and it became clear that the reason for inviting me was that no competent native successor was in sight. I took over a class of students in their formative years, and I am happy to say that many have remained my friends and are now important mathematicians in their own right.

I cannot honestly say that I was happy in Zürich. The postwar era was not a good time for a stranger to take root in Switzerland. The whole nation, although spared from war, was in a state of suspension, with, understandably, quite a bit of xenophobia in the lower classes. My wife and I did not feel welcome outside the circle of our immediate colleagues.

I was therefore very pleased when in 1946 I was asked if I would like to return to Harvard. My Swiss colleagues and the "Erziehungsdirektion des Kantons Zürich" did their best to persuade me to stay, but I was convinced that for the sake of my mathematical career I should go to America, and I was consoled when I learned that my teacher Rolf Nevanlinna had agreed to become my successor. This arrangement turned out to be a great success both for Nevanlinna and for the university.

In the fall of 1946 I took up my duties at Harvard, where I was to stay until my retirement in 1977 and as emeritus to this day. My relationship with Harvard, with both the administration and my friends in the department, has been singularly happy. I have enjoyed the many excellent students Harvard has had to offer, many of whom I have watched become leaders in my own or some other field. Harvard has also offered me an optimal milieu for my research, and it has been a source of great satisfaction that the Mathematics Department has been able to maintain the high standards that have always been its hallmark.

The annotations to my papers will serve as a running commentary to my scientific activity. I take this opportunity to thank Birkhäuser Boston, Inc., for their willingness to publish these collected works, and above all my friend Gian-Carlo Rota for taking the initiative and supervising the editing of this book.

Carathéodory's Report on Ahlfors' work for Fields Medal at ICM in Oslo 1936

Lars Valerian Ahlfors wurde in Helsingfors am 18. April 1907 als Sohn des Professors für Maschinenbau an der dortigen Technischen Hochschule Axel Ahlfors geboren. Nachdem er die »Nya svenska samskolan« absolviert hatte, wurde er im Mai 1924 an der Universität Helsingfors immatrikuliert, wo er am 5. März 1928 »Filosofie kandidat« und am 20. November 1930 »Filosofie licentiat« wurde. Bei der solemnen Promotion an derselben Universität wurde er am 31. Mai 1932 zum »Filosofie magister« und »Filosofie doktor« promoviert.

Von 1929—1933 war er Lektor der Mathematik an der Åbo Akademie und seit Juni 1933 ist er Adjunkt der Mathematik an der Universität Helsingfors. Im akademischen Jahre 1935—1936 ist er beurlaubt worden, um an der Universität Harvard Vorlesungen zu halten. Im Jahre 1936 wurde er Mitglied der Societas Scientiarum Fennica.

Wiederholt haben ihn Studienreisen ins Ausland geführt: im Winter 1928—1929 war er in Zürich, im Frühjahr 1929 in Paris, im Sommer 1931 in Göttingen und im Sommer 1934 in München.

Das akademische Lehrjahr 1931—1932 verbrachte er in Paris als »International Research Council Fellow« der »Rockefeller Foundation«.

Ahlfors ist einer der glänzendsten Vertreter der berühmten Finnländischen Funktionentheoretischen Schule, die von Ernst Lindelöf gegründet, seit dreißig Jahren so viele wichtige Beiträge der Wissenschaft geschenkt und so zahlreiche große Mathematiker hervorgebracht hat. Er ist Schüler von Ernst Lindelöf und von Rolf Nevanlinna. Unter den Auspizien des letzteren ist seine Dissertation entstanden und die Ideen und Theorien von R. Nevanlinna haben weiterhin seine ganze Entwickelung beeinflußt.

Ausgegangen ist er von der Theorie der meromorphen Funktionen, die er von Anbeginn mit originellen Methoden befruchtet hat. So konnte er einmal einen wichtigen Satz von Henri Cartan für seine Zwecke benutzen, und seine Dissertation »Untersuchungen zur Theorie der konformen Abbildungen und der ganzen Funktionen« (1930) enthält eine Methode für die Behandlung der konformen Abbildung des verallgemeinerten Streifens mit beliebig gekrümmten Rändern, die sofort bei ihrem Erscheinen Aufsehen erregt hat.

Die reifste Frucht, die das Studium der meromorphen Funktionen durch Ahlfors gezeitigt hat, findet man in seiner Arbeit »Über eine Methode in der Theorie der meromorphen Funktionen« (1935). Wenn man diese Arbeit liest, so weiß man nicht, was mehr zu bewundern ist: die Kunst, mit der Ahlfors die ganze, große Nevanlinnasche Theorie mit wunderbarer Klarheit auf nur 14 Seiten auseinanderzusetzen versteht, oder die geniale Intuition von Rolf Nevanlinna, der zu einer Zeit, wo die geometrischen Zusammenhänge noch ganz versteckt lagen, nur solche Begriffe entdeckt, ausgebildet und benutzt hat, die später die einfachste geometrische Deutung erfahren sollten.

Bei dem Entschluß des Komitees, eine der Fields-Medaillen Lars Ahlfors zuzuerkennen, ist aber vor allen eine andere Richtung seiner Arbeiten ins Gewicht gefallen. Das Studium der Riemannschen Flächen der Umkehrfunktionen von ganzen oder meromorphen Funktionen hat Ahlfors dazu geführt, allgemeinere Eigenschaften von Überlagerungsflächen zu betrachten. Es stellte sich allmählich heraus, daß die fertige, unberandete Überlagerungsfläche gewisse ganz allgemeine Eigenschaften besitzt, die man nur beobachten kann, wenn man sie gewissermaßen in statu nascendi als Grenze von berandeten, immer weiter um sich greifenden Überlagerungsflächen betrachtet. Bisher war diese Entstehungsweise einer Riemannschen Fläche, die Hermann Amandus Schwarz die Methode des Ölflecks nannte, nur für die Zwecke der konformen Abbildung und der Uniformisierungstheorie benutzt worden. Es blieb Ahlfors vorbehalten, asymptotische Gesetze aufzustellen, in welche der Inhalt der behandeten Approximationsflächen, die Länge des Randes dieser Flächen und die charakteristische Zahl der Grundfläche eingehen.

Die Arbeit »Zur Theorie der Überlagerungsflächen« (1935), die diesen Untersuchungen gewidmet ist, kann als das erste Kapitel eines neuen Zweiges der Analysis angesehen werden, für welchen vielleicht der Name »metrische Topologie« der geeignete ist. Die Überlagerungsflächen sollen nämlich alle auf einer Grundfläche liegen, die gewisse metrische Eigenschaften hat.

Den Gebieten auf der Grundfläche und ihren Rändern werden in allgemeinster Weise gewisse Zahlen zugeordnet, die, wenn man sie als Inhalt und Länge deutet, einer isoperimetrischen Ungleichheit genügen sollen. Diese scheinbar so wenig sagende Bedingung ist aber hinreichend, um die Ahlforsche Theorie abzuleiten.

Die schönsten Anwendungen dieser Resultate beziehen sich auf die Theorie der quasikonformen Abbildungen. Es zeigt sich z. B., daß der Picardsche Satz schon für quasikonforme Abbildungen der dreifach punktierten Kugel gilt, also eigentlich als Satz der metrischen Topologie angesehen werden muß. Wenn nämlich eine Überlagerungsfläche der Kugel auf die ganze euklidische Ebene quasikonform abgebildet wird, so gibt es in der Ebene immer Folgen von konzentrischen Kreisen, deren

Bilder auf der Kugel folgende Eigenschaften haben: die Länge des Randes dieser sphärischen mehrblättrigen Fläche dividiert durch ihren Gesamtflächeninhalt, konvergiert gegen Null. Nach der Ahlforschen Theorie ist es aber unmöglich, diese Bedingung zu befriedigen wenn drei Punkte der Kugel von der Figur nicht überdeckt werden dürfen.

Noch merkwürdiger ist der Ahlforsche Scheibensatz, von dem ein partikulärer Fall folgendermaßen ausgesprochen werden kann: Die ganze xy-Ebene werde auf eine Riemannsche Fläche der uv-Ebene quasikonform irgendwie abgebildet. Man wähle drei beliebige einfach zusammenhängende schlichte Gebiete der uv-Ebene, die außerhalb einander liegen. Dann hat die Riemannsche Fläche die Eigenschaft, daß mindestens das eine ihrer Blätter auch das eine dieser drei beliebigen, aber fest gewählten Gebiete enthalten muß. Oder mit anderen Worten: es gibt in der xy-Ebene mindestens ein Gebiet, das eineindeutig auf das eine oder andere der drei Gebiete der uv-Ebene abgebildet wird. Daß man mit zwei Gebieten keinen derartigen Satz aufstellen kann, zeigt schon die konforme Abbildung, die durch die Gleichung $w = \sin z$ hervorgerufen wird.

Est ist leider unmöglich, die ganze Tragweite der Ahlforschen Theorie in wenigen Worten verständlich zu machen; nach diesen wenigen Proben sieht man aber schon, daß es sich um eine ganz erstklassige Leistung handelt.

Ahlfors' Address as Honorary President at opening of ICM in Berkeley 1986

I accept this great honor with a good conscience because I consider myself a link between this International Congress and the one in 1936, fifty years ago, the occasion on which the Fields Medals were given for the first time. I understand that my only duty here will be the pleasure of handing out the Fields Medals and the Nevanlinna Prize.

At that time the circumstances were quite different; the idea of the medals had been approved in Zurich in 1932, but there had been no publicity about it and when I arrived in Oslo I did not know that the Medal had become a reality, and if I had known it I would not have considered myself the right candidate. As a matter of fact, I had not been told anything officially until I entered the room where the opening ceremony would take place, but there I was shown a place somewhere in front, and I may have had my suspicions. Well, I had more than that. I had been warned beforehand by somebody who by mistake congratulated me a day before. But up to that point it had been a secret at least officially, even to myself. There was no tradition to go by and no protocol to follow. As was mentioned here, two medals were given, one to me and one to Jesse Douglas, who was then at MIT while I happened to be a visiting lecturer at Harvard. In that way it so happened that both medals went to Cambridge, Massachusetts. Unfortunately Douglas could not accept his medal in person because according to the Congress record he was too tired. I don't know, but maybe he had good reason to be tired after a long and strenuous journey. I would not expect that to happen today. His medal was then accepted by Norbert Wiener as representative of MIT.

There are two traditions that go back to the very beginning. In the first place, the Committee to select the winners should consist of the top brass of contempory mathematics. In 1936 the members of that Committee were G.D. Birkhoff, Caratheodory, Elie Cartan, Severi, and Takagi. Truly I would call that a panel of Olympian heroes. And I think that this tradition has been continued at subsequent Congresses. The other tradition is that the works of the winners should be commented on by prominent persons in the field. In 1936 both prizes were explained by Caratheodory.

As was mentioned there was no Congress until 1950, fourteen years later. On that occasion, which took place at Harvard, the medals were given to Atle Selberg and Laurent Schwartz, both known and admired by all mathematicians. From then on the Fields Medals have become more and more prestigious and it is a safe bet that many dream of getting it. Whether true or not that the existence of the medal has contributed to the phenomenal growth of mathematics both in quantity and quality during the last fifty years must remain anybody's guess.

Today it is safe to congratulate the winners in advance and I use this occasion to offer them my sincerest compliments to their success. I share their feeling of pride and accomplishment and I know that their continued success is guaranteed. I also share the disappointment of the many who may feel that they have been passed by. I wish them better luck next time or, if there is not a next time, that posterity will prove them right and the Committee wrong. Thank you.

AN EXTENSION OF SCHWARZ'S LEMMA*

BY
LARS V. AHLFORS

I. The fundamental inequality

1. To every neighborhood on a Riemann surface there is given a map onto a region of the complex plane. For any two overlapping neighborhoods the corresponding maps are directly conformal.† We agree to denote points on the surface by $\mathfrak{w}$, corresponding values of the local complex parameter by w.

We introduce a Riemannian metric of the form

$$ds = \lambda \, | dw |, \tag{1}$$

where the positive function λ is supposed to depend on the particular parameter chosen, in such a way that ds becomes invariant. The metric is regular if λ is of class C_2. In this paper we shall, without mentioning it further, allow λ to become zero, although such points are of course singularities of the metric.

It is well known that the Gaussian curvature of the metric (1) is given by

$$K = - \lambda^{-2} \Delta \log \lambda, \tag{2}$$

and that this expression remains invariant under conformal mappings of the w-plane. We are interested in the case of a metric with negative curvature, bounded away from zero. It is convenient to choose the upper bound of the curvature equal to -4. From (2) it follows that the corresponding λ satisfies the condition

$$\Delta \log \lambda \geqq 4\lambda^2. \tag{3}$$

When we set $u = \log \lambda$ this is equivalent to

$$\Delta u \geqq 4e^{2u}. \tag{4}$$

The hyperbolic metric of the unit circle $|z| < 1$ is defined by

$$d\sigma = (1 - | z |^2)^{-1} | dz | \tag{5}$$

and has the constant curvature -4.

2. Consider now an analytic function $\mathfrak{w} = f(z)$ from the circle $|z| < 1$ to a Riemann surface W. The analyticity is expressed by the fact that every local parameter w is an analytic function of z. To a differential element dz corresponds an element dw whose length does not depend on the direction of dz. The corresponding value of $ds = \lambda |dw| = \lambda_z |dz|$ is therefore uniquely de-

* Presented to the Society, September 8, 1937; received by the editors April 1, 1937.

† For the definition of a Riemann surface see T. Radó, *Über den Begriff der Riemannschen Fläche*, Acta Szeged, vol. 2 (1925).

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termined, and we have $\lambda_s = \lambda |w'(z)|$. It is also seen that $u = \log \lambda_s$ satisfies the condition (4) whenever the given metric has a curvature ≤ -4. An exception has to be made for the possible zeros of λ_s, corresponding to the zeros of λ and $w'(z)$.

Theorem A. *If the function $w = f(z)$ is analytic in $|z| < 1$, and if the metric (1) of W has a negative curvature ≤ -4 at every point, then the inequality*

$$ds \leq d\sigma \tag{6}$$

will hold throughout the circle.

Proof: Choose an arbitrary $R < 1$ and set $v = \log R(R^2 - |z|^2)^{-1}$ for $|z| < R$. We note that $\Delta v = 4e^{2v}$ and consequently

$$\Delta(u - v) \geq 4(e^{2u} - e^{2v}). \tag{7}$$

Let us denote by E the open point set in $|z| < R$ for which $u > v$. It is clear that E cannot contain any zeros of λ_s. Hence (7) is valid and shows that $u - v$ is subharmonic in E. It follows that $u - v$ can have no maximum in E and must approach its least upper bound on a sequence tending to the boundary of E. But E can have no boundary points on $|z| = R$, for v becomes positively infinite as z tends to that circle, and at interior boundary points we must have $u - v = 0$, by continuity. A contradiction is thus obtained, unless E is vacuous. The inequality $u \leq v$ consequently subsists for all points with $|z| < R$, and letting R tend to 1 we find $u \leq -\log(1 - |z|^2)$ at all points. This is equivalent to (6).

If W is the unit circle and ds its hyperbolic metric, Theorem A is simply the differential form of Schwarz's lemma given by Pick.*

3. Several generalizations of the theorem just proved suggest themselves at once. Since the only thing we need is to prevent the function $u - v$ from having a maximum in E, it is obvious that the assumptions on λ can be considerably weakened, without affecting the validity of the argument. We shall give below two such generalizations which are found to be particularly useful for the applications.

Theorem A1. *Let λ be continuous and such that at every point, either (a) the second derivatives of $u = \log \lambda$ are continuous and satisfy (4), or (b) it is possible to find two opposite directions n', n'' for which $\partial u/\partial n' + \partial u/\partial n'' > 0$. Then the statement of the previous theorem is still true.*

Opposite directions in the w-plane correspond to opposite directions in the z-plane. At a maximum of $u - v$ we have $\partial u/\partial n \leq \partial v/\partial n$ in any direction, when-

* An account of all questions related to Schwarz's lemma will be found in R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, 1936, pp. 45–58.

ever the directional derivative exists. For opposite directions $\partial v/\partial n' + \partial v/\partial n'' = 0$; hence $\partial u/\partial n' + \partial u/\partial n'' \leq 0$ in case of a maximum. It follows that no maximum can be attained in points satisfying condition (b).

We shall call $ds' = \lambda'|dw|$ a supporting metric of $ds = \lambda|dw|$ at the point w_0 if: (1) $\lambda' = \lambda$ at w_0, (2) λ' is defined and $\leq \lambda$ in a neighborhood of w_0.

THEOREM A2. *Suppose that λ is continuous, and that it is possible to find a supporting metric, satisfying* (4), *at every point of W. Then the inequality* (6) *still holds.*

If $u - v > 0$ at z_0, then $u' - v$ will also be positive, and consequently subharmonic, in a neighborhood of z_0.* A maximum of $u - v$ will *a fortiori* be a maximum of $u' - v$. Hence $u - v$ can have no maximum in E.

II. SCHOTTKY'S THEOREM

4. As a first application we prove Schottky's theorem with definite numerical bounds.

THEOREM B. *If $f(z)$ is analytic and different from* 0 *and* 1 *in $|z| < 1$, then*

$$\log |f(z)| < \frac{1+\theta}{1-\theta}(7 + \overset{+}{\log} |f(0)|) \tag{8}$$

for $|z| \leq \theta < 1$.†

Let $\zeta_1 = \zeta_1(w)$ map the region outside of the segment (0, 1) onto the exterior of the unit circle, so that $w = \infty$ corresponds to $\zeta_1 = \infty$, $w = 1$ to $\zeta_1 = 1$, and $w = 0$ to $\zeta_1 = -1$. We also set $\zeta_2(w) = \zeta_1(w^{-1})$ and $\zeta_3(w) = \zeta_1(1 - w)$. Clearly these functions define similar maps of the regions outside of the segments $(1, \infty)$ and $(-\infty, 0)$. Explicitly, $\zeta_1(w)$ is obtained from the equation

$$\zeta_1 + \zeta_1^{-1} = 4w - 2. \tag{9}$$

We introduce the coordinates $\rho_1 = |w|$, $\rho_2 = |w - 1|$ and divide the plane into regions Ω_1: $\rho_1 \geq 1$, $\rho_2 \geq 1$; Ω_2: $\rho_1 \leq 1$, $\rho_1 \leq \rho_2$; Ω_3: $\rho_2 \leq 1$, $\rho_2 \leq \rho_1$. The metric

$$ds_i = \frac{|d \log \zeta_i|}{2(4 + \log |\zeta_i|)} = \lambda_i |dw| \tag{10}$$

---

* u' corresponds to λ' as u to λ.

† Schottky's original theorem was purely qualitative. Numerical relations have been studied at great length, notably by Ostrowski (*Studien über den Schottky'schen Satz*, Basel, 1931, and *Asymptotische Abschätzung des absoluten Betrags einer Funktion, die die Werte* 0 *und* 1 *nicht annimmt*, Commentarii Mathematici Helvetici, vol. 5 (1933)), but no simple inequality comparable with (8) has ever been proved.

Added in proof: Numerical bounds of the same order of magnitude are found by A. Pfluger, *Über numerische Schranken im Schottky'schen Satz*, Commentarii Mathematici Helvetici, vol. 7 (1935). His proof depends on the use of modular functions, while ours is strictly elementary.

is readily recognized as the hyperbolic metric of a half-plane with the constant curvature -4. Computing the derivatives $\zeta_i'(w)$ we find

$$
\begin{aligned}
\lambda_1^{-1} &= 2(\rho_1\rho_2)^{1/2}(4 + \log|\zeta_1|),\\
\lambda_2^{-1} &= 2\rho_1\rho_2^{1/2}(4 + \log|\zeta_2|),\\
\lambda_3^{-1} &= 2\rho_2\rho_1^{1/2}(4 + \log|\zeta_3|).
\end{aligned}
\tag{11}
$$

We now set $ds = \lambda|dw|$ with $\lambda = \lambda_i$ in Ω_i. This metric is regular and satisfies condition (3) except at the singular points $0, 1, \infty$ and on the lines separating the regions Ω_i. On these lines λ is still continuous, as seen from (11) and the relations between ζ_1, ζ_2, and ζ_3.

Next we wish to show that condition (b) in Theorem A1 holds on the singular lines. We consider the arc $\rho_1 = 1$, $\rho_2 > 1$ and choose n', n'' as the outer and inner normals of the circle. The required condition is

$$\frac{\partial}{\partial n'}\log\lambda_1 + \frac{\partial}{\partial n''}\log\lambda_2 = \frac{\partial}{\partial n'}\log\frac{\lambda_1}{\lambda_2} > 0.$$

From (11) we obtain

$$\frac{\partial}{\partial n'}\log\frac{\lambda_1}{\lambda_2} = \frac{1}{2} - \frac{\dfrac{\partial}{\partial n'}\log\left|\dfrac{\zeta_1}{\zeta_2}\right|}{4 + \log|\zeta_1|},$$

which is also equal to

$$\frac{1}{2} - 2(4 + \log|\zeta_1|)^{-1}\frac{\partial\Phi_1}{\partial\phi},$$

where $\Phi_1 = \arg\zeta_1$, $\phi = \arg w$. For Φ_1 we have the simple relation $\cos\Phi_1 = \rho_1 - \rho_2$, which for $\rho_1 = 1$ becomes $\cos\Phi_1 = 1 - 2\sin\phi/2$. Differentiating we find

$$\frac{\partial\Phi_1}{\partial\phi} = \frac{1}{2}\left(1 + \csc\frac{\phi}{2}\right)^{1/2},$$

and by use of the inequalities $\pi/3 \leqq \phi \leqq 5\pi/3$, $|\zeta_1| > 1$, we are finally led to the desired result,

$$\frac{\partial}{\partial n'}\log\frac{\lambda_1}{\lambda_2} > \frac{1}{2} - \frac{3^{1/2}}{4} > 0.$$

By symmetry, the same must be true for the arc $\rho_2 = 1$, $\rho_1 > 1$. The transformation $w' = (1 - w)^{-1}$ takes Ω_1 into Ω_2 and Ω_2 into Ω_3. Since the function λ is invariant under the transformation we conclude at once that condition (b) will hold also on the line separating Ω_2 and Ω_3.

From Theorem A1 we can now conclude that $w = f(z)$ satisfies the differential inequality $\lambda|dw| \leq (1-|z|^2)^{-1}|dz|$. Integrating, we find that the shortest distance between the points $f(0)$ and $f(z)$, $|z|=\theta$, measured in the metric $ds=\lambda|dw|$, cannot exceed $[\log(1+\theta)/(1-\theta)]/2$.

The shortest path between the circles $\rho_1=m$ and $\rho_1=M$, where $M>m\geq 2$, is a segment of the negative real axis, whose length is found to be

$$\frac{1}{2}\log\frac{4+\log|\zeta_1(-M)|}{4+\log|\zeta_1(-m)|}.$$

To simplify we introduce the lower and upper bounds $|\zeta_1(-M)|\geq 4M$, $|\zeta_1(-m)|\leq 5m$. Setting $M=|f(z)|$ and m equal to the greater of the numbers $|f(0)|$ and 2 we obtain

$$4+\log 4M \leq \frac{1+\theta}{1-\theta}(4+\log 5m).$$

Here $\log 5m \leq \log 10+\overset{+}{\log}|f(0)| < 3+\overset{+}{\log}|f(0)|$ and we find

$$4+\log 4M < \frac{1+\theta}{1-\theta}(7+\overset{+}{\log}|f(0)|)$$

which is stronger than (8).

III. Bloch's theorem

5. Let $w=f(z)$ be analytic in $|z|<1$ with $|f'(0)|=1$. Let $B'=B'(f)$ be the l.u.b. of the radii of all simple (*schlicht*) circles contained in the Riemann surface W generated by $f(z)$. Bloch's theorem is $B=\min B'>0$. Landau has proved $B>.396$.* Grunsky and Ahlfors proved in a recent paper $B<.472$.†

We show that the method developed in this paper gives an immediate proof of Bloch's theorem with a better lower bound for B. For an arbitrary point $\mathfrak{w}$ on W let $\rho(\mathfrak{w})$ denote the radius of the largest simple circle of center $\mathfrak{w}$ contained in W. It is clear that $\rho(\mathfrak{w})$ is continuous, and equal to zero only at the branch-points. We introduce the metric $ds=\lambda|dw|$ with

$$\lambda = \frac{A}{2\rho^{1/2}(A^2-\rho)} \qquad (\rho=\rho(\mathfrak{w})) \tag{12}$$

and w denoting the variable of the function plane (not the uniformizing variable). A is a constant satisfying the preliminary condition $A^2>B'$.

In the neighborhood of a branch-point a we have $\rho=|w-a|$. Let n be the multiplicity of a; then $w_1=(w-a)^{1/n}$ is a uniformizing variable, and

* E. Landau, *Über den Blochschen Satz und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929).

† L. V. Ahlfors and H. Grunsky, *Über die Blochsche Konstante*, Mathematische Zeitschrift, vol. 42 (1937). The result was found independently by R. M. Robertson.

the corresponding λ_1 is determined from $\lambda_1|dw_1| = \lambda|dw|$. We obtain $\lambda_1 = n\rho^{1/2-1/n}/2(A^2-\rho)$, and it is seen at once that the metric is regular in case $n=2$ and that λ_1 becomes zero in case $n>2$.

We wish to apply Theorem A2 and therefore look for a supporting metric satisfying the requirements of that theorem. For a regular point $\mathfrak{w}_0$ the surrounding circle of radius $\rho(\mathfrak{w}_0)$ must pass through at least one singularity $\mathfrak{b}$ which is either a branch-point or a boundary point for the surface. We set $\rho' = |w-b|$ and define $\lambda' = A/[2\rho'^{1/2}(A^2-\rho')]$. This metric has the curvature -4 for it is obtained from the hyperbolic metric of a circle by means of the transformation $w' = w^{1/2}$. In all points of our circle we have $\rho \leqq \rho'$ by the definition of ρ. The inequality $\lambda' \leqq \lambda$ is therefore satisfied in a neighborhood of $\mathfrak{w}_0$ if the function $t^{1/2}(A^2-t)$ increases for $t \leqq \rho(\mathfrak{w}_0)$. Under this condition λ' will be a supporting function of λ, for at the center w_0 we have $\lambda' = \lambda$. The function $t^{1/2}(A^2-t)$ is increasing as long as $t<A^2/3$. Consequently all the conditions in Theorem A2 are fulfilled if we suppose that $A^2>3B'$.

Apply the theorem with $z=0$. Using the condition $|dw/dz|_{z=0} = 1$ we get

$$A \leqq 2\rho_0^{1/2}(A^2 - \rho_0), \tag{13}$$

where ρ_0 is the radius of the largest simple circle with center at the image of $z=0$. The function in the right member of (13) is increasing, and we can replace ρ_0 by B' obtaining $A \leqq 2B'^{1/2}(A^2-B')$. Letting A tend to $(3B')^{1/2}$ we finally get $B' \geqq 3^{1/2}/4$. This implies that Bloch's constant $B \geqq 3^{1/2}/4 > .433$.

On the other side, if we insert $A^2 = (3B')^{1/2}$ in (13), lower and upper bounds for ρ_0 in terms of B' can be found.

6. Landau has considered a closely related constant L. Let $L' = L'(f)$ be the l.u.b. of the radii of all circles in the w-plane contained in the projection of W, that is, whose values are taken by the function $w=f(z)$, $|f'(0)| = 1$. L is defined as the minimum of all such L'. Clearly, $L \geqq B$.

The method employed above is immediately applicable if we choose $\lambda = (2\rho \log C/\rho)^{-1}$. This metric is regular at all branch-points, and when we replace ρ by the distance ρ' from a fixed boundary point, the curvature becomes -4. In order that the function λ' thus obtained be a supporting function it is sufficient that $t \log C/t$ is increasing. This is true for $t<Ce^{-1}$. We therefore choose $C>eL'$, obtaining the inequality $1 \leqq 2L' \log C/L'$ as above. Letting C tend to eL' we find $L' \geqq 1/2$ and hence $L \geqq 1/2$.

This lower bound is the best known. It shows in particular that $L>B$.*

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* In the other direction R. M. Robinson has proved $L<.544$. This result has not been published.

QUASICONFORMAL REFLECTIONS

BY

LARS V. AHLFORS

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Let L be a Jordan curve on the Riemann sphere, and denote its complementary components by Ω, Ω^*. Suppose that there exists a sense-reversing quasiconformal mapping λ of the sphere onto itselfs which maps Ω on Ω^* and keeps every point on L fixed. Such mappings are called quasiconformal reflections. Our purpose is to study curves L which permit quasiconformal reflections.

Let U denote the upper and U^* the lower halfplane. Consider a conformal mapping f of U on Ω and a conformal mapping f^* of U^* on Ω^*. Evidently, $f^{*-1}\lambda f$ defines a quasiconformal mapping of U on U^* which induces a monotone mapping $h = f^{*-1}f$ of the real axis on itself. It is not quite unique, for we may replace f by fS and f^* by f^*S^* where S and S^* are linear transformations with real coefficients and positive determinant. This replaces h by $S^{*-1}hS$ which we shall say is equivalent to h. Observe that h, or rather its equivalence class, does not depend on λ. It is also unchanged if we replace the triple (Ω, L, Ω^*) by a conformally equivalent triple $(T\Omega, TL, T\Omega^*)$ where T is a linear transformation.

The mapping f of U has a quasiconformal extension to the whole plane, namely by the mapping with values $\lambda f(\bar{z})$ for $z \in U^*$. It is known that quasiconformal mappings carry nullsets into nullsets. Therefore L has necessarily zero area.

From this we may deduce that h determines Ω uniquely up to conformal equivalence. In fact, let f_1, f_1^* be another pair of conformal mappings on complementary regions, and suppose that $f_1^{*-1}f_1 = f^{*-1}f$ on the real axis. For a moment, let us write F for the mapping given by $f(z)$ in U and by $\lambda f(\bar{z})$ in U^*, and let F_1 have the corresponding meaning. The mapping $H = F_1^{-1}f_1^*f^{*-1}F$ is defined in U^* and reduces to the identity on the real axis. We extend it to the whole plane by setting $H(z) = z$

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in U. Then $F_1 H F^{-1}$ is a quasiconformal mapping. It reduces to $f_1 f^{-1}$ in Ω and to $f_1^* f^{*-1}$ in Ω^*. It is thus conformal, except perhaps on L. But a quasiconformal mapping which is conformal almost everywhere is conformal. Hence $f_1 = Tf$ where T is a linear transformation.

What are the properties of h? A necessary condition is that h can be extended to a quasiconformal mapping of U on U^*, namely to $f^{*-1}\lambda f$. This condition is also sufficient. To prove it, let g be a quasiconformal mapping of U on U^* with boundary values h. The function $g^*(z) = g(\bar{z})$, defined in U^*, has weak derivatives which satisfy an equation

$$g^*_{\bar{z}} = \mu g^*_z$$

with $|\mu| \le k < 1$ (k constant). Set $\mu = 0$ in U. Consider the equation

$$F_{\bar{z}} = \mu F_z$$

for the extended μ. An important theorem (see [1]), sometimes referred to as the generalized Riemann mapping theorem, asserts the existence of a solution F which is a homeomorphic mapping of the sphere. Because z is a solution in U and g^* a solution in U^* it is possible to write $F = f$ in U, $F = f^* g^*$ in U^*, where f and f^* are conformal mappings. Clearly, $\Omega = f(U)$ and $\Omega^* = f^*(U^*)$ are quasiconformal reflections of each other.

To sum up, we have established a correspondence between equivalence classes of boundary correspondences h, conformal mappings f, and curves L which permit a quasiconformal reflection. It is a natural program to try to characterize the possible h, f and L in a more direct way. For boundary correspondences h this problem has been solved; we shall have occasion to recall the solution.

In Part I we solve the corresponding problem for L. It turns out that the curves which permit a quasiconformal reflection can be characterized by a surprisingly simple geometric property. (Partial results in this direction have been obtained by M. Tienari whose paper [7] came to my attention only when this article was already written.)

We have been less successful with the mappings f, but in Part II we show, at any rate, that the mappings f form an open set. To understand the meaning of this, we observe that the mappings equivalent to f are of the form TfS. To account for T we replace f by its Schwarzian derivative $\varphi = \{f, z\}$. The Schwarzian of fS is $\varphi(S)S'^2$, and to eliminate S it is indicated to consider φdz^2 in its role of quadratic differential.

If f is schlicht in U, Nehari [6] has shown that $|\varphi| y^2 \le \frac{3}{2}$. We take the least upper bound of $|\varphi| y^2$ to be a norm of φ. In the linear space of quadratic differentials with finite norm, let Δ be the set of all φ whose corresponding f is schlicht and has a quasiconformal extension. We are going to show that Δ is an open set. For the significance of this result in the theory of Teichmüller spaces we refer to the companion article of L. Bers [4] in the next issue of this journal.

Part I

1. In 1956 A. Beurling and the author derived a neccessary and sufficient condition for a boundary h to be the restriction of a quasiconformal mapping of U on itself (or on its reflection U^*). This work is an essential preliminary for what follows.

We recall the main result. Without loss of generality it may be assumed that $h(\infty) = \infty$. Then h admits a quasiconformal extension if and only if it satisfies a ϱ-condition, namely an inequality

$$\varrho^{-1} < \frac{h(x+t) - h(x)}{h(x) - h(x-t)} < \varrho, \tag{1}$$

which is to be fulfilled for all real x, t and with a constant $\varrho \neq 0, \infty$. More precisely, if h has a K-quasiconformal extension, then (1) holds with a $\varrho(K)$ that depends only on K, and if (1) holds, then h has a $K(\varrho)$-quasiconformal extension.

The necessity follows from the simple observation that the quadruple $(x-t, x, x+t, \infty)$ with cross-ratio 1 must be mapped on a quadruple with bounded cross-ratio. The sufficiency requires an explicit construction. We set $w(z) = u + iv$ with

$$\left.\begin{aligned} u(z) &= \int_0^1 [h(x+ty) + h(x-ty)]\,dt, \\ v(z) &= \int_0^1 [h(x+ty) - h(x-ty)]\,dt. \end{aligned}\right\} \tag{2}$$

It is proved in [2] that $w(z)$ is $K(\varrho)$-quasiconformal.

I am indebted to Beurling for the very important observation that the mapping (2) is also quasi-isometric, in the sense that corresponding noneuclidean elements of length have a bounded ratio. This condition can be expressed by

$$\left.\begin{aligned} |w_z| &< C(\varrho)\frac{v}{y}, \\ |w_{\bar z}| &> C(\varrho)^{-1}\frac{v}{y}, \end{aligned}\right\} \tag{3}$$

where $C(\varrho)$ depends only on ϱ. The proof is an immediate verification based on the estimate given in Lemma 6.5 of the cited paper.

2. Let L be a Jordan curve through ∞ which admits a quasiconformal reflection. The complementary regions determined by L are denoted by Ω, Ω^*, and the reflection is written as $z \to z^*$. We assume that the reflection is K-quasiconformal.

Constants which depend only on K will be denoted by $C(K)$, with or without subscripts. In different connections $C(K)$ may have different values. We emphasize that $C(K)$ is not allowed to depend on L.

The shortest distance from a point z to L will be denoted by $\delta(z)$.

LEMMA 1. *The following estimates hold for all z in the plane and all z_0 on L:*

(a) $$C(K)^{-1} \leq \left|\frac{z^* - z_0}{z - z_0}\right| \leq C(K)$$

(b) $$\frac{|z - z^*|}{\delta(z)} \leq C(K)$$

(c) $$C(K)^{-1} \leq \frac{\delta(z^*)}{\delta(z)} \leq C(K).$$

Proof. If the cross-ratio of a quadruple has absolute value ≤ 1, then the cross-ratio of the image points under a K-quasiconformal mapping has an absolute value $\leq C(K)$. This assertion is contained in [1], Lemma 16. It is a rather elementary result.

If $|z^* - z_0| \leq |z - z_0|$ we can apply the above remark to (z^*, z, z_0, ∞) and conclude that $|z - z_0| \leq C(K)|z^* - z_0|$. Symmetrically, $|z - z_0| \leq |z^* - z_0|$ implies $|z^* - z_0| \leq C(K)|z - z_0|$. In all circumstances (a) follows.

From (a) we obtain

$$|z - z^*| \leq (C(K) + 1)|z - z_0| = C_1(K)|z - z_0|$$

and (b) follows when $|z - z_0| = \delta(z)$. Since $\delta(z^*) \leq |z - z^*|$ the second inequality (c) follows from (b), and the first is true by symmetry.

2. We introduce now the noneuclidean metrics $ds = \varrho(z)|dz|$ in Ω and Ω^*. Explicitly, if $z = z(\zeta)$ is a conformal map of $|\zeta| < 1$ on Ω we set

$$\varrho(z)|dz| = (1 - |\zeta|^2)^{-1}|d\zeta|.$$

The classical estimates

$$\delta(z) \leq \varrho(z)^{-1} \leq 4\delta(z) \tag{4}$$

follow by use of Schwarz' lemma and Koebe's one-quarter theorem.

LEMMA 2. *If L passes through ∞ and permits a K-quasiconformal reflection, then it also permits a $C(K)$-quasiconformal reflection with the additional property that corresponding euclidean line elements satisfy*

$$C_1(K)^{-1}|dz| < |dz^*| < C_1(K)|dz|. \tag{5}$$

Proof. As shown in the introduction, the given K-quasiconformal reflection induces a K-quasiconformal mapping of U on U^* with a boundary correspondence h. This h must satisfy a $\varrho(K)$-condition of type (1). The Beurling-Ahlfors construction permits us to replace the mapping of U on U^* with a $C(K)$-quasiconformal mapping with the same boundary values, in such a way that it satisfies condition (3). It follows that the corresponding reflection about L is $C(K)$-quasiconformal and satisfies

$$C_1(K)^{-1}\varrho(z)|dz| < \varrho(z^*)|dz^*| < C_1(K)\varrho(z)|dz|.$$

Use of (4) and (c) leads to the desired inequality (5).

3. We are now ready to characterize the curves L in purely geometric form:

THEOREM 1. *A Jordan curve L through ∞ permits a quasiconformal reflection if and only if there exists a constant C such that*

$$\overline{P_1P_2} < C\cdot\overline{P_1P_3} \tag{6}$$

for any three points P_1, P_2, P_3 on L which follow each other in this order.

Again, there is a more precise statement to the effect that C depends only on the K of the reflection, and vice versa. If L does not pass through ∞ condition (6) must be replaced by

$$\overline{P_1P_2}:\overline{P_1P_3} < C(\overline{P_4P_2}:\overline{P_4P_3}),$$

where (P_1, P_3) separates (P_2, P_4).

4. *Proof of the necessity*. We follow the segment P_1P_3 from P_1 to its last intersection with the subarc $P_2P_1\infty$ of L, and from there to the first intersection P_3' with the arc $P_2P_3\infty$. If $P_1P_2 > P_1P_3$ it is geometrically evident that

$$\overline{P_1P_2}:\overline{P_1P_3} < \overline{P_1'P_2}:\overline{P_1'P_3'}.$$

Therefore we may assume from the beginning that $\overline{P_1P_3}$ has only its endpoints on L. For definiteness, we suppose that the inner points lie in Ω.

By Lemma 2 there exists a quasiconformal reflection which multiplies lengths at most by a factor $C(K)$. Hence P_1 and P_3 can be joined in Ω^* by an arc γ^* of length $< C(K) \cdot \overline{P_1P_3}$. The Jordan curve formed by $\overline{P_1P_3}$ and γ^* separates P_2 from ∞. Hence γ^* intersects the extension of $\overline{P_1P_2}$ over P_2 and we conclude that $\overline{P_1P_2} <$ length of $\gamma^* < C(K) \cdot \overline{P_1P_3}$.

5. *Proof of the sufficiency*. We shall use the notations

$$\alpha = \text{arc } P_2P_3, \quad \alpha' = \text{arc } P_1P_2,$$
$$\beta = \text{arc } P_1\infty, \quad \beta' = \text{arc } P_3\infty.$$

Denote by $d(\alpha, \beta)$ and $d^*(\alpha, \beta)$ the extremal distances of α and β with respect to Ω and Ω^* respectively. With similar notations for α', β' one has the relations

$$d(\alpha, \beta)\, d(\alpha', \beta') = d^*(\alpha, \beta)\, d^*(\alpha', \beta') = 1.$$

In a conformal mapping of Ω on the halfplane U with ∞ corresponding to ∞, let P_1, P_2, P_3 be mapped on x_1, x_2, x_3. It is evident that $d(\alpha, \beta) = 1$ if and only if $x_3 - x_2 = x_2 - x_1$. Furthermore, the ratio $|x_3 - x_2| : |x_2 - x_1|$ is bounded away from 0 and ∞ if and only if this is true of $d(\alpha, \beta)$. Consequently, in order to prove that the boundary correspondence induced by L satisfies (1) it is sufficient to show that $d(\alpha, \beta) = 1$ implies $K(C)^{-1} < d^*(\alpha, \beta) < K(C)$.

Two elementary estimates are needed. We show first that $d(\alpha, \beta) = 1$ implies

$$\overline{P_1P_2} : \overline{P_2P_3} < C^2 e^{2\pi}. \tag{7}$$

Indeed, it follows from (6) that the points of β are at distance $> C^{-1} \cdot \overline{P_1P_2}$ from P_2 while the points of α have distance $< C \cdot \overline{P_2P_3}$ from P_2. If (7) were not true, α and β would be separated by a circular annulus whose radii have the ratio $e^{2\pi}$. In such an annulus the extremal distance between the circles is 1, and the comparison principle for extremal lengths would yield $d(\alpha, \beta) > 1$, contrary to hypothesis. Hence (7) must hold. If P_1 and P_3 are interchanged we have in the same way

$$\overline{P_2P_3} : \overline{P_1P_2} < C^2 e^{2\pi}. \tag{8}$$

Consider points $Q_1 \in \alpha$, $Q_2 \in \beta$. By repeated application of (6)

$$\overline{Q_1Q_2} > C^{-1}\, \overline{Q_1P_1} > C^{-2}\, \overline{P_1P_2}$$

and with the help of (8) we conclude that the shortest distance between α and β is $> C^{-4} e^{-2\pi}\, \overline{P_2P_3}$. To simplify notations, write $d = \overline{P_2P_3}$, $M_1 = Cd$, $M_2 = C^{-4} e^{-2\pi} d$. Because of (6), all points on α are within distance M_1 from P_2.

We recall that the definition of extremal length implies

$$d^*(\alpha,\beta) > \frac{(\inf \int_\gamma \varrho |dz|)^2}{\iint_{\Omega^*} \varrho^2 dx\, dy},$$

where the infimum is with respect to all arcs γ that join α and β within Ω^*, and ϱ is any positive function for which the right-hand side has a meaning. We choose $\varrho = 1$ in a circular disk with center P_2 and radius $M_1 + M_2$, $\varrho = 0$ outside of that disk. Then $\int_\gamma \varrho |dz| > M_2$ for all curves γ. Indeed, this is so whether γ stays within the disk or contains a point on its circumference. We conclude that

$$d^*(\alpha,\beta) > \frac{1}{\pi}\left(\frac{M_2}{M_1+M_2}\right)^2 = \pi^{-1}(1 + C^6 e^{2\pi})^{-2}.$$

The same inequality, applied to α', β', yields an upper bound for $d^*(\alpha,\beta)$, and our proof of Theorem 1 is complete.

Part II

1. In the introduction we saw that the boundary correspondences h give rise to conformal mappings f, and with these we associated their Schwarzian derivatives $\varphi = \{f, z\}$. The set of all such φ was denoted by Δ. We formulate a precise definition:

The set Δ consists of all functions φ, holomorphic in U, such that the equation $\{f, z\} = \varphi$ has a solution f which can be extended to a schlicht quasiconformal mapping of the whole plane.

Our purpose is to prove:

THEOREM 2. *Δ is an open subset of the Banach space of holomorphic functions with norm* $\|\varphi\| = \sup |\varphi(z)| y^2$.

We know already that all $\varphi \in \Delta$ have norm $\leq \frac{3}{2}$. It will follow that the norms are in fact strictly less than $\frac{3}{2}$.

2. It is a known result that Δ contains a neighborhood of the origin ([3], [5]). As an illustration of the method we shall follow it is nevertheless useful to include a proof.

LEMMA 3. *Δ contains all functions φ with* $\|\varphi\| < \frac{1}{2}$.

Proof. Let η_1 and η_2 be linearly independent solutions of the differential equation

$$\eta'' = -\tfrac{1}{2}\varphi\eta. \tag{9}$$

normalized by $\eta_1'\eta_2 - \eta_2'\eta_1 = 1$. It is well known that $f = \eta_1/\eta_2$ satisfies $\{f, z\} = \varphi$. Observe that f may be meromorphic with simple poles, and that $f' \neq 0$ at all other points!

It is to be shown that f is schlicht and has a quasiconformal extension. To construct the extension we form

$$F(z) = \frac{\eta_1(z) + (\bar{z} - z)\eta_1'(z)}{\eta_2(z) + (\bar{z} - z)\eta_2'(z)} \quad (z \in U). \tag{10}$$

Because $\eta_1'\eta_2 - \eta_2'\eta_1 = 1$ the numerator and denominator cannot vanish simultaneously. If the denominator vanishes we set $F = \infty$, and local assertions about F will apply to $1/F$.

A simple computation which makes use of (9) gives

$$F_z/F_{\bar{z}} = \tfrac{1}{2}(z - \bar{z})^2 \varphi(z).$$

Under the assumption $\|\varphi\| < \frac{1}{2}$ we conclude that F is quasiconformal and sense-reversing. The mapping $z \to F(\bar{z})$ is quasiconformal and sense-preserving in U^*.

Our intention is to show that

$$f(z) = \begin{cases} f(z) \text{ in } U \\ F(\bar{z}) \text{ in } U^* \end{cases} \tag{11}$$

gives the desired extension. To see this it is sufficient to know that f can be extended to the real axis by continuity, that the extended function is locally schlicht at points of the real axis, and that it tends to a limit for $z \to \infty$. Indeed, f will then be locally schlicht everywhere, and by a familiar reasoning is must be globally schlicht.

The missing information is easy to supply under strong additional conditions. We suppose that φ is analytic on the real axis, including ∞, where φ shall have a zero of order ≥ 4 (this means that the quadratic differential φdz^2 is regular at ∞). It is immediate that f and F agree on the real axis, and that they are real-analytic in the closed half-planes. It follows easily that f is locally schlicht. At ∞ the assumption implies that equation (9) has solutions whose power series expansions begin with 1 and z respectively. Hence

$$\eta_1 = a_1 z + b_1 + O(|z|^{-1}),$$
$$\eta_2 = a_2 z + b_2 + O(|z|^{-1})$$

with $a_1 b_2 - a_2 b_1 = 1$. Substitution in (10) shows that

$$F(z) = \frac{a_1 \bar{z} + b_1 + O(|z|^{-1})}{a_2 \bar{z} + b_2 + O(|z|^{-1})}$$

and therefore f and F have the same limit a_1/a_2 as $z \to \infty$.

To prove the lemma without additional assumptions we use an approximation method. We can find a sequence of linear transformations S_n such that the closure of $S_n U$ is contained in U and $S_n z \to z$ for $n \to \infty$. Take $\varphi_n(z) = \varphi(S_n z)\, S_n'(z)^2$. It follows by Schwarz' lemma that $\|\varphi_n\| < \|\varphi\|$. Moreover, φ_n is analytic on the real axis and has at least a fourth order zero at ∞. Consequently, there exist quasiconformal mappings f_n, holomorphic with $\{f_n, z\} = \varphi_n$ in U, with uniformly bounded dilatation. A subsequence of the f_n converges to a limit function f which is itself schlicht and quasiconformal, and which satisfies $\{f, z\} = \varphi$ in U. This completes the proof.

With suitable normalizations it is possible to arrange that $f_n \to f$, the mapping defined by (11).

3. The method of the preceding proof can be carried over to the general case, although with some significant modifications.

Suppose that $\varphi_0 \in \Delta$ and $\{f_0, z\} = \varphi_0$. We may assume that f_0 maps U on a region Ω whose boundary L passes through ∞, and we know that L admits a quasiconformal reflection $w \to w^* = \lambda(w)$. We choose λ in accordance with Lemma 2.

If $\|\varphi - \varphi_0\| < \varepsilon$ and $\{f, z\} = \varphi$ the identity

$$\{f, z\} = \{f, f_0\}\, f_0'^2 + \{f_0, z\}$$

yields

$$|\{f, f_0\}|\, |f_0'|^2 y^2 < \varepsilon.$$

The non-euclidean metric in Ω is given by

$$\varrho(w)\,|dw| = \frac{|dz|}{2y},$$

and if we write $\bar{f} = f f_0^{-1}$ we obtain

$$|\{\bar{f}, w\}| < 4\varepsilon\varrho(w)^2. \tag{12}$$

If ε is sufficiently small it is to be proved that $\bar{f}$ is schlicht and has a quasiconformal extension.

We set $\bar{\varphi} = \{\bar{f}, w\}$ and $\bar{f} = \eta_1/\eta_2$ where η_1, η_2 are normalized solutions of

$$\eta'' = -\tfrac{1}{2}\bar{\varphi}\eta.$$

In close analogy with (10) we form

$$F(w) = \frac{\eta_1(w) + (w^* - w)\,\eta_1'(w)}{\eta_2(w) + (w^* - w)\,\eta_2'(w)},$$

where $w \in \Omega$ and $w^* = \lambda(w)$. Computation gives

$$\frac{F_w}{F_{\bar{w}}} = \frac{\lambda_w}{\lambda_{\bar{w}}} + \frac{\bar{\varphi}(w - w^*)^2}{2\lambda_{\bar{w}}}. \tag{13}$$

Here $|\lambda_w/\lambda_{\bar{w}}| < k < 1$ because λ is quasiconformal. To estimate the second term we have first, by (12), Lemma 1 (b) and (4),

$$|\bar{\varphi}||w - w^*|^2 < 4\varepsilon C^2.$$

On the other hand, $|\lambda_{\bar{w}}|$ stays away from 0, for Lemma 2 gives

$$C^{-1}|dw| \leq |dw^*| \leq 2|\lambda_{\bar{w}}||dw|.$$

We conclude that $|F_w/F_{\bar{w}}| \lesssim k' < 1$ provided that ε is sufficiently small.

4. We wish to show that

$$f = \begin{cases} f(w) & \text{in } \Omega \\ F(w^*) & \text{in } \Omega^* \end{cases}$$

is schlicht and quasiconformal. Again, the proof is easy under strong assumptions. This time we assume that L is an analytic curve, that $\bar{\varphi}$ is analytic on L and that it has a fourth order zero at ∞. It is clear that we can prove f to be a quasiconformal homeomorphism exactly as in the proof of Lemma 3.

To complete the proof, let $\zeta = \omega(w)$ be a conformal mapping of Ω on $|\zeta| < 1$. Let Ω_n be the part of Ω that corresponds to $|\zeta| < r_n$, L_n its boundary. Here $\{r_n\}$ is a sequence which converges to 1.

A quasiconformal reflection λ_n across L_n can be constructed as follows: If $r_n^2 < |\omega(w)| < r_n$ we define $\lambda_n(w)$ so that $\omega(w)$ and $\omega(\lambda_n(w))$ are mirror images with respect to $|\zeta| = r_n$. If $|\omega(w)| \leq r_n^2$ we find w_n so that $\omega(w_n) = r_n^{-2}\omega(w)$ and choose $\lambda_n(w) = \lambda(w_n)$. The definitions agree when $|\omega(w)| = r_n^2$, and L_n is kept fixed. The dilatation of λ_n is no greater than the maximum dilatation of λ.

After a harmless linear transformation which throws a point on L_n to ∞ the part of the theorem that has already been proved can be applied to Ω_n. It is to be observed that $\varrho_n > \varrho$ where $\varrho_n|dw|$ is the noneuclidean metric in Ω_n. Therefore $\bar{\varphi}$ satisfies

$$|\bar{\varphi}| < 4\,\varepsilon\varrho_n(w)^2$$

with the same ε as before. Hence there exists a quasiconformal mapping f_n of the whole plane which agrees with f on Ω_n and whose dilatation lies under a fixed bound. A subsequence of the f_n tends to a limit mapping f which is schlicht, quasiconformal, and equal to f in Ω. The theorem is proved.

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FINITELY GENERATED KLEINIAN GROUPS.*

By Lars V. Ahlfors.

1. Introduction. 1.1. Let Γ be a group of linear transformations

$$Az = \frac{az+b}{cz+d}, \quad ad-bc=1$$

of the extended complex plane. A point z_0 is called a *limit point* of Γ if it is an accumulation point of points Az_1, $A \in \Gamma$, some some z_1. The set of limit points will be denoted by $\Sigma(\Gamma)$, or by Σ when it is clear to what group we are referring.

$\Sigma(\Gamma)$ is a closed set, and invariant under Γ; we denote its complement by $\Omega = \Omega(\Gamma)$. As soon as $\Omega(\Gamma)$ is not empty we say that Γ is discontinuous, and $\Omega(\Gamma)$ is its *set of discontinuity*. It is easy to classify all groups for which Σ is void, consists of a single point, or of two points. All other discontinuous groups will be called *Kleinian groups*. In other words, a Kleinian group is a discontinuous group with more than two limit points. In this paper Γ will always denote a Kleinian group.

For Kleinian groups the images Az_1, $A \in \Gamma$, of any point z_1 accumulate toward all of Σ. It is an immediate consequence that Σ is a nowhere dense perfect set, and that every nonvoid closed invariant set includes Σ.

A Kleinian group which leaves a circle invariant is said to be Fuchsian. The set of limit points lies on the invariant circle. If Σ is the whole circle the Fuchsian group is of the first kind. If not, Σ is nowhere dense on the invariant circle, and the group is of the second kind.

The fixpoints of non-elliptic transformations in Γ lie on Σ. There are always infinitely many hyperbolic or loxodromic transformations, and their fixpoints are dense on Σ. The fixpoints in Ω belong to elliptic transformations, and they are isolated.

1.2. In order to study the action of Γ on Ω we consider the quotient space $S = \Omega/\Gamma$. It has a natural complex structure such that the projection map $\pi: \Omega \to S$ is holomorphic. Thus, the components S_i of S are Riemann

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surfaces. We shall write $\Omega_i = \pi^{-1}S_i$. In general, the Ω_i are not connected, and we denote the components of Ω_i by Ω_{ij}.

With any Kleinian group Γ we have thus associated the decompositions $\Omega = \cup \Omega_i = \cup \Omega_{ij}$ and $S = \cup S_i$. Each Ω_i is invariant under the full group Γ, and the boundary of Ω_i is all of Σ. The components Ω_{ij} are ramified covering surfaces of S_i whose branch points are elliptic fixpoints.

The projections of the branch points will be denoted by p_k. They are isolated, and with each p_k there is associated an integer $m_k \geqq 2$, the order of the corresponding elliptic transformation. We say that S has a signature $I = \{p_k, m_k\}$. S together with its signature is denoted by $S(I)$, and $S_i(I)$ has a similar meaning.

We shall say that S is of *finite type* if there are finitely many components S_i and points p_k, and if each S_i can be obtained from a compact surface by omission of a finite number of points. When this is so it is convenient to let S_i denote the compact surface, and to regard the omitted points as points p_k with $m_k = \infty$. It will be clear from the context when S_i should be interpreted in this manner.

1.3. The main result to be proved in this article is the following statement:

If Γ is finitely generated, then S is of finite type.

This is known for Fuchsian groups, but to the author's knowledge there is no proof in the published literature. A proof is contained in a manuscript of Nielsen and Fenchel, where the result is obtained at the end of a penetrating study of Fuchsian groups. Recently, L. Bers has found a short proof by variational methods, the same that have proved so useful in the theory of Teichmüller spaces.[1]

My task has been to show that Bers' method can be extended to arbitrary Kleinian groups. The main difficulty was the correct formulation and proof of a crucial lemma, which is to be considered the core of this paper. We have formulated it in two parts, as Lemma 8 and Lemma 9.

In the last section we prove some results on functions groups, that is, groups with an invariant region. These were the results that gave the initial impetus to this study.

1.4. Perhaps of greater interest are the theorems I have not been able to prove. For instance, it seems very likely that the set Σ for any finitely generated group Γ has zero area, but I cannot prove it.

[1] Unpublished.

Any Kleinian group can be extended to a corresponding group of Möbius transformations in the Poincaré half-space. If Γ is finitely generated it seems plausible that the Poincaré group has a fundamental polyhedron with only a finite number of vertices on Σ. Again, I cannot prove this, but if this is so I am able to show that Σ has indeed zero measure.

2. The Poincaré metric. 2.1. Each component Ω_{ij} carries a Poincaré metric $ds = \rho\,|\,dz\,|$ with constant negative curvature equal to -1, and such that Ω_{ij} is complete in this metric. The metric is unique, and therefore

$$\rho(Az)\,|\,A'(z)\,| = \rho(z) \tag{2.1}$$

for all $A \in \Gamma$ which map Ω_{ij} onto itself. We shall consider ρ on all of Ω, defined in each Ω_{ij}. Then (2.1) holds for all $A \in \Gamma$.

The projection π induces a corresponding metric $\bar\rho\,|\,d\zeta\,|$ on S. Here ζ is a local parameter on S, and $\bar\rho$ is determined by $\bar\rho\,|\,d\zeta\,| = \rho\,|\,dz\,|$. At the branch points $d\zeta/dz = 0$, and we see that $\bar\rho$ becomes infinite at the points p_k.

If S_i is compact the Gauss-Bonnet formula yields

$$\chi(S_i) + \sum_{p_k \in S_i} \left(1 - \frac{1}{m_k}\right) = \frac{1}{2\pi}\int_{\Omega_i/\Gamma} \rho^2\, dx dy, \tag{2.2}$$

where $\chi(S_i)$ is the Euler characteristic. The formula remains valid for any $S_i(I)$ of finite type, provided that $\chi(S_i)$ is the characteristic of the compact surface and the infinite signatures are included in the sum. To prove (2.2) for this case requires knowledge of the asymptotic behavior of $\bar\rho$ at the punctures. If the omitted point corresponds to $\zeta = 0$ this behavior is given by

$$\log \bar\rho = -\log|\,\zeta\,| - \log\log(1/|\,\zeta\,|) + O(1), \tag{2.3}$$

a classical result.

An immediate consequence of (2.2) is that

$$\chi(S_i) + \sum_{p_k \in S_i} \left(1 - \frac{1}{m_k}\right) > 0. \tag{2.4}$$

This rules out certain possibilities. Without insisting on the details, we conclude that S_i cannot be a sphere with certain low signatures, nor can it be a torus without signature.

A more careful analysis of the possible cases shows that (2.4) can be sharpened to

$$\chi(S_i) + \sum_{S_i} \left(1 - \frac{1}{m_k}\right) \geqq 1/42.$$

This is important because it shows that S *is of finite type if and only if it:*

total Poincaré area is finite. The conclusion uses the elementary fact that a single surface with finite Poincaré area is necessarily of finite type.

2.2. We show next that the points p_k with $m_k = \infty$ correspond to parabolic fixpoints. Since the result does not require S to be of finite type, let us merely assume that $S_i = \bar{S}_i - \{p_k\}$ where $\bar{S}_i$ is a Riemann surface and $p_k \in \bar{S}_i$.

LEMMA 1. *There exists a parabolic transformation $A_k \in \Gamma$ with the following property: If the linear transformation U is such that $U^{-1}A_kUz = z+1$, then Ω_i contains a set $\{z \mid \operatorname{Im} Uz > c\}$ and $\pi(z) \to p_k$ when $z \in \Omega_i$ approaches the fixpoint $z_k = U\infty$ in such a way that $\operatorname{Im} Uz \to \infty$.*

We remark that A_k can of course be replaced by any conjugate transformation $B^{-1}A_kB$ with $B \in \Gamma$, in which case z_k is replaced by Bz_k. The set $\{z \mid \operatorname{Im} Uz > 0\}$ is a disk or a halfplane whose boundary passes through z_k. It follows that there are at most two p_k which correspond to the same z_k.

For the proof of the lemma we may regard p_k as the center of a punctured disk $\bar{\Delta} \subset S_i$, represented by $0 < |\zeta| < 1$, and we may assume that no point in $\bar{\Delta}$ has a signature. Let Δ be a component of $\pi^{-1}(\bar{\Delta})$. It is clear that the cover transformations of Δ over $\bar{\Delta}$ are generated by a single transformation $A \in \Gamma$ which is not elliptic.

If A is hyperbolic or loxodromic we can take it to be of the form $Az = kz$, $|k| \neq 1$. In terms of $s = \log \zeta$ the projection π induces a one-one correspondence $z = z(s)$ between the halfplane $\operatorname{Re} s < 0$ and Δ which satisfies $z(s + 2\pi i) = kz(s)$. Because 0 is a limit point it is not in Δ, and therefore the distance from $z(s)$ to the boundary of Δ is at least $|z(s)|$. With this information Koebe's one-quarter theorem easily yields

$$|z'(s)|/|z(s)| \leqq 2/|\operatorname{Re} s|.$$

Integration from s to $s + 2\pi i$ along a vertical line gives

$$|\log |k|| \leqq 4\pi/|\operatorname{Re} s|.$$

But this is absurd, for $|\operatorname{Re} s|$ can be chosen arbitrarily large.

We conclude that A is parabolic, for instance $Az = z + 1$. In this case $z(s + 2\pi i) = z(s) + 1$ and it is elementary to deduce that

$$\exp(2\pi i z) = \zeta + a_0 + a_1/\zeta + \cdots,$$

from which

$$z(s) = \frac{s}{2\pi i} + O(|\zeta|^{-1}) \tag{2.5}$$

and, for later use,

(2.6) $$dz/d\zeta = (2\pi i)^{-1} + O(|\zeta|^{-2}).$$

It is immediate from (2.5) that Ω_ζ contains a halfplane $y > c$, and that $y \to \infty$ implies $\pi(z) \to p_k$. This is the assertion of the lemma when U is the identity, and the general case follows by application to the group $U^{-1}\Gamma U$.

2.3. We shall use (2.6) to derive the asymptotic behavior of ρ in terms of the uniformizer ζ. Because $\rho\,|dz| = \bar{\rho}\,|d\zeta|$ we obtain, with the help of (2.3),

(2.7) $$\log\rho = -\log\log(1/|\zeta|) + O(1).$$

In the derivation of this formula we have assumed that U is the identity, but it is easy to see that it remains valid as soon as the fixpoint is at ∞.

In case of a finite fixpoint we make the transformation $z = Uw$. The Poincaré metric in $U^{-1}\Omega$ is given by $\rho_0(w)\,|dw| = \rho(z)\,|dz|$. It is for ρ_0 that (2.7) is valid, while (2.5) is applicable to w in the place of z. We have $\log\rho = \log\rho_0 - \log|U'(w)| = \log\rho_0 + 2\log|w| + O(1)$, for since $U\infty$ is finite, $w^2U'(w)$ has a finite limit when $w \to \infty$. It follows that

$$\log\rho = -\log\log(1/|\zeta|) + 2\log|\log\zeta| + O(1).$$

The expression on the right is not single-valued, but we may conclude that

(2.8) $$\log\rho = \log\log(1/|\zeta|) + O(1)$$

when z approaches the fixpoint in an interior angle.

2.4. The coefficient of the Poincaré metric decreases when the region increases (this is essentially Schwarz' lemma). Denote the euclidean distance from z to Σ by $\delta(z)$. If we compare Ω with the circle about z of radius $\delta(z)$ it follows at once that $\rho(z) \leqq 2\delta(z)^{-1}$. In case Ω_U is simply connected and does not contain ∞ an opposite inequality can be obtained by use of Koebe's one-quarter theorem. One finds under these circumstances $\rho(z) \geqq \frac{1}{2}\delta(z)^{-1}$.

In the general case we use the majorization $\Omega \subset \Omega_0$, where Ω_0 is the sphere punctured at three points $z_0, z_1, z_2 \in \Sigma$. We have $\rho \geqq \rho_0$, and ρ_0 can be estimated by means of (2.3). It follows that

$$\log\rho(z) \geqq -\log|z - z_0| - \log\log(1/|z - z_0|) - O(1)$$

in a neighborhood of z_0. It is not hard to see that the remainder is uniformly bounded when z_0 varies but stays away from z_1, z_2. If Σ is assumed to be compact we conclude that

(2.9) $$\log\rho \geqq -\log\delta - \log\log(1/\delta) - O(1)$$

for sufficiently small δ.

3. Quadratic differentials. 3.1. In all that follows a dominant role is played by the quadratic differentials on Ω. A quadratic differential is a holomorphic function ϕ on Ω which satisfies the functional equations

$$\phi(Az)A'(z)^2 = \phi(z) \tag{3.1}$$

for all $A \in \Gamma$. More accurately, the quadratic differential is the invariant expression $\phi\, dz^2$, but it is convenient to use the same term for the coefficient ϕ.

The projection map induces a corresponding quadratic differential on S, namely by the relation $\phi\, dz^2 = \bar{\phi}\, d\zeta^2$. At a point p_k with finite signature, let the projection be given by $\zeta = z^m$ in local coordinates. Then $\bar{\phi} = O(|\zeta|^{-2+2/m})$, and since $\bar{\phi}$ is single-valued it has at most a simple pole at p_k. Conversely, if $\bar{\phi}$ has at most a simple pole, then ϕ is holomorphic.

As soon as S is not compact the class of all quadratic differentials on Ω is too extensive to be useful. We shall consider the following restriction:

$$(Q) \qquad \int_{\Omega/\Gamma} |\phi|\, dxdy < \infty,$$

$$(Q^*) \qquad |\phi| = o(\rho^2).$$

Observe that the integral in (Q) is meaningful because of (3.1). The quadratic differentials which satisfy these conditions form linear spaces which we denote by $Q(\Gamma)$ and $Q^*(\Gamma)$ respectively.

In terms of $\bar{\phi}$ condition (Q) becomes

$$\int_S |\bar{\phi}|\, d\xi d\eta < \infty. \tag{3.2}$$

Similarly, (Q^*) yields $|\bar{\phi}| = O(\bar{\rho}^2)$. In both cases $\bar{\phi}$ has at most a simple pole at points with infinite signature. In fact, for Q this follows directly from (3.2), and for Q^* we may use (2.3) to deduce that $\bar{\phi} = O(|\zeta|^{-2})$ in terms of a local variable, and hence that the singularity is at most a simple pole.

We shall let $\bar{Q}(S)$ and $\bar{Q}^*(S)$ denote the spaces formed by all $\bar{\phi}$ for $\phi \in Q(\Gamma)$ and $\phi \in Q^*(\Gamma)$ respectively. If S is of finite type these spaces are identical, and we conclude:

LEMMA 2. *If S is of finite type the spaces $Q(\Gamma)$ and $Q^*(\Gamma)$ are identical and of finite dimension.*

The dimension over the complex numbers is $3g + n - 3k$ where g is the sum of the genera of the surfaces S_i, n is the total number of points with a signature, and k is the number of components. It follows from condition (2.4) that there are non-zero quadratic differentials on each S_i.

3.2. We have just proved one part of the following result, which is particularly important for our purposes:

THEOREM 1. *The space $Q(\Gamma)$ is finite dimensional if and only if S is of finite type.*

If there are infinitely many S_i, all of finite type, it follows from the remark above that $Q(\Gamma)$ has infinite dimension. Suppose now that one S_i is not of finite type. If S_i has infinite genus there exist infinitely many linearly independent square integrable first order differentials θ_j on S_i. If they are multiplied by a fixed one, θ_0, the $\theta_0\theta_j$ are linearly independent quadratic differentials, and they are integrable. Assume next that S_i has finite genus. Then it can be imbedded in a compact Riemann surface $\bar{S}_i$. If $\bar{S}_i - S_i$ is an infinite point set we can find infinitely many quadratic differentials on $\bar{S}_i$ with distinct simple poles on $\bar{S}_i - S_i$. Their restrictions to S_i are integrable and linearly independent. Finally, if $\bar{S}_i - S_i$ is a finite set there must be infinitely many p_k on S_i, and we can find infinitely many quadratic differentials with distinct poles among the p_k. Thus, whenever S is not of finite type, $Q(\Gamma)$ has infinite dimension.

4. Beltrami differentials. 4.1. We regard $Q(\Gamma)$ as a Banach space with the norm

$$\| \phi \| = \int_{\Omega/\Gamma} | \phi | \, dxdy$$

and $Q^*(\Gamma)$ as one with the norm

$$\| \phi \|_* = \sup | \phi | \rho^{-2}.$$

One of our aims is to determine the conjugate space of $Q(\Gamma)$. Several characterizations will be obtained, and their comparison will contribute to our knowledge of Kleinian groups.

4.2. Choose an arbitrary $\phi_0 \in Q(\Gamma)$, not identically zero, and define a measure on $S = \Omega/\Gamma$ by $dm = | \phi_0 | \, dxdy$. Evidently, a quadratic differential ϕ belongs to $Q(\Gamma)$ if and only if ϕ/ϕ_0 is of class L^1 with respect to this measure. Therefore, every linear functional on $Q(\Gamma)$ can be represented as

$$\int_{\Omega/\Gamma} \phi_0^{-1} | \phi_0 | \beta \, dxdy$$

where β is a bounded measurable function on S. We may define β as an automorphic function on Ω, and we find that $\nu = \phi_0^{-1}|\phi_0|\beta$ satisfies the equations

$$\nu(Az)\bar{A}'(z)/A'(z) = \nu(z) \tag{4.1}$$

for all $A \in \Gamma$. Such functions are called *Beltrami differentials*; we emphasize that ν has to be bounded and measurable. We have proved that every linear functional on $Q(\Gamma)$ is of the form

$$\int_{\Omega/\Gamma} \phi\nu \, dxdy \tag{4.2}$$

where ν is a Beltrami differential.

4.3. Let $B(\Gamma)$ be the linear space of Beltrami differentials, normed by $\|\nu\|_\infty$, and let $N(\Gamma)$ denote the subspace of those ν which satisfy

$$\int_{\Omega/\Gamma} \phi\nu \, dxdy = 0 \tag{4.3}$$

for all $\phi \in Q(\Gamma)$. Our result can be stated as follows:

LEMMA 3. *The conjugate space of* $Q(\Gamma)$ *can be identified with* $B(\Gamma)/N(\Gamma)$.

Clearly, any characterization of $N(\Gamma)$ will therefore be a characterization of the conjugate space of $Q(\Gamma)$.

5. The Fuchsian case. 5.1. Many problems in the theory of Kleinian groups can be reduced to the corresponding problem for Fuchsian groups. We shall therefore begin with a study of the Fuchsian case.

To emphasize the distinction, Fuchsian groups will be denoted by the letter G. We assume that G is discontinuous and acts on the unit disk $D = \{z \mid |z| < 1\}$, but it need not be of the first kind. We shall constantly disregard the outside of D. Therefore, we shall set $S(G) = D/G$, and the notations $Q(G)$ and $Q^*(G)$ will refer to quadratic differentials in D. The corresponding remark applies to the spaces $B(G)$ and $N(G)$. Observe that $\rho(z) = 2(1-|z|^2)^{-1}$.

We shall make frequent use of the known representation formula [3]

$$\phi(\zeta) = 3\pi^{-1}\int_D \phi(z)(1-|z|^2)^2(1-\bar{z}\zeta)^{-4}\, dxdy \tag{5.1}$$

[3] L. Ahlfors, "Some remarks on Teichmüller's space of Riemann surfaces," *Annals of Mathematics*, vol. 74, No. 1, 1961, p. 176.

which is valid as soon as ϕ is holomorphic in D and

(5.2) $$\int_D |\phi|(1-|z|^2)^2\,dxdy < \infty.$$

This condition is obviously fulfilled when $\phi \in Q^*(G)$.

The identity

(5.3) $$\int_D |1-\bar{z}\zeta|^{-4}\,dxdy = \pi(1-|\zeta|^2)^{-2}$$

will also be needed.

5.2. With any Beltrami differential $\nu \in B(G)$ we associate

(5.4) $$T\nu(\zeta) = 3\pi^{-1}\int_D \bar{\nu}(z)(1-\bar{z}\zeta)^{-4}\,dxdy.$$

A simple computation shows that $T\nu$ is a quadratic differential, and by use of (5.3) we conclude that $T\nu \in Q^*(G)$. In the opposite direction, $\phi \in Q^*(G)$ can be mapped on $T^*\phi = \bar{\phi}(1-|z|^2)^2 = 4\bar{\phi}\rho^{-2} \in B(G)$. Formula (5.1) takes the form $TT^*\phi = \phi$, or $TT^* = I$. The operator T^* is of course meaningful also when $\phi \in Q(G)$, but then we can no longer assert that $T^*\phi$ belongs to $B(G)$.

As an abbreviation we introduce

(5.5) $$\langle \phi, \nu \rangle = \int_{D/G} \phi\nu\,dxdy,$$

and we use this notation as soon as the integral on the right is absolutely convergent. One verifies at once that

(5.6) $$\langle \phi_1, T^*\phi_2 \rangle = \langle \phi_2, T^*\phi_1 \rangle^-$$

as soon as $\phi_1 \in Q(G)$, $\phi_2 \in Q^*(G)$ or vice versa (the bar stands for the complex conjugate).

Even more important is the following identity:

LEMMA 4. $\langle \phi, T^*T\nu \rangle = \langle \phi, \nu \rangle$ *for* $\phi \in Q(G)$, $\nu \in B(G)$.

Both sides of the identity represent absolutely convergent integrals, and by (5.6) the left hand side can be replaced by $\langle T\nu, T^*\phi \rangle^-$. By virtue of (5.1) the equality $\langle T\nu, T^*\phi \rangle^- = \langle \phi, \nu \rangle$ is equivalent to

$$\int_{D/G} \phi(\zeta)(1-|\zeta|^2)^2\,d\xi d\eta \int_D \nu(z)(1-\bar{z}\zeta)^{-4}\,dxdy$$
$$= \int_D \phi(\zeta)(1-|\zeta|^2)^2\,d\xi d\eta \int_{D/G} \nu(z)(1-\bar{z}\zeta)^{-4}\,dxdy,$$

an identity which is proved by subdivision into "fundamental regions" and subsequent changes of the integration variables. The absolute convergence justifies the procedure.

5.3. Let $A(D)$ be the space of holomorphic functions which are integrable over D. For $F \in A(D)$ the Poincaré series

$$\Theta F(z) = \sum_{A \in G} F(Az) A'(z)^2 \tag{5.7}$$

converges and represents an element of $Q(G)$. In fact, one obtains

$$\| \Theta F \| \leqq \int_D |F| \, dxdy. \tag{5.8}$$

It is also easy to verify that

$$\langle \Theta F, \nu \rangle = \int_D F\nu \, dxdy. \tag{5.9}$$

5.4. We now deduce the most important result in this direction, which is due to L. Bers:[8]

THEOREM 2. *The mapping* $T: B(G) \to Q^*(G)$ *is a surjection with kernel* $N(G)$. *Therefore, the conjugate space of* $Q(G)$ *can be identified with* $Q^*(G)$.

That the mapping is onto follows from $TT^* = I$. Another way of expressing Lemma 4 is to say that $\nu - T^*T\nu \in N(G)$. Therefore, $T\nu = 0$ implies $\nu \in N(G)$. To prove the converse, let F be the function $3\pi^{-1}(1 - z\bar{\zeta})^{-4}$ with fixed $\zeta \in D$. The definition (5.4) of $T\nu$ together with (5.9) gives

$$T\nu(\zeta)^- = \langle \Theta F, \nu \rangle,$$

and hence $\nu \in N(G)$ yields $T\nu = 0$ as asserted.

Specifically, we have shown that every coset of $N(G)$ in $B(G)$ contains an element $T^*\phi$ with $\phi \in Q^*(G)$, namely for $\phi = T\nu$. This is of course the only element with this property, for $T^*\phi \in N(G)$ implies $\phi = TT^*\phi = 0$.

5.5. As an immediate corollary we obtain:

LEMMA 5. *Every* $\phi \in Q(G)$ *can be written in the form* ΘF *with* $F \in A(D)$.

Indeed, it follows from (5.8) that Θ maps $A(D)$ onto a closed subspace of $Q(G)$. If it were not all of $Q(G)$ we could find $\phi \in Q^*(G)$, not identically zero, such that $\langle \Theta F, T^*\phi \rangle = 0$ for all $F \in A(D)$. By (5.9) this gives

[8] Oral communication.

$$\int_D FT^*\phi\, dxdy = 0.$$

We choose again $F(z) = 3\pi^{-1}(1 - z\bar{\zeta})^{-4}$ and obtain $\phi(\zeta) = 0$, contrary to assumption.

5.6. There is a generalization of Lemma 5 which will be of importance when we pass to the case of Kleinian groups. Let G_0 be an arbitrary subgroup of G. For any $\phi_0 \in Q(G)$ we construct

$$(5.10) \qquad \Theta_0\phi_0(z) = \sum \phi_0(Az)A'(z)^2$$

where the summation is now over a set of representatives of the left cosets of G_0, one from each coset. Because ϕ_0 is a quadratic differential with respect to G_0 the choice of representatives is irrelevant, and one finds that $\| \Theta_0\phi_0 \| \leq \| \phi_0 \|$. Equation (5.9) is replaced by

$$(5.11) \qquad \langle \Theta_0\phi_0, \nu \rangle = \langle \phi_0, \nu \rangle,$$

valid for all $\nu \in B(G)$. It is to be observed that the first inner product is over D/G, the second over D/G_0.

LEMMA 6. *Every* $\phi \in Q(G)$ *can be written as* $\Theta_0\phi_0$ *with* $\phi_0 \in Q(G_0)$.

Again, if this were not true we could find $\phi \in Q^*(G)$, $\phi \neq 0$, such that $\langle \Theta_0\phi_0, T^*\phi \rangle = 0$ for all $\phi_0 \in Q(G)$. According to (5.11) this gives $\langle \phi_0, T^*\phi \rangle = 0$ for all ϕ_0, and hence $\phi = 0$, a contradiction.

6. The general case. 6.1. We return to the case of an arbitrary Kleinian group Γ, except that we shall assume ∞ to be a limit point. This minor restriction has the advantage that the Poincaré series

$$(6.1) \qquad \Theta F(z) = \sum_{A \in \Gamma} F(Az)A'(z)^2$$

remains holomorphic for any $F \in A(\Omega)$, that is, whenever F is holomorphic and integrable in Ω. It is clear that ΘF enjoys the same properties as in the Fuchsian case.

We shall show that the conclusion in Theorem 2 as well as Lemma 5 remain valid. As far as the conjugate space is concerned it is quite clear that we may restrict our attention to a single component Ω_y of Ω and replace Γ by the subgroup Γ_y which leaves Ω_y invariant. It amounts to the same thing if we assume from the beginning that Ω is an invariant region with respect to Γ, but not necessarily the whole set of discontinuity.

Let $z = \omega(\zeta)$ be a conformal mapping of $D = \{\zeta \mid |\zeta| < 1\}$ onto the universal covering surface of Ω. We introduce a Fuchsian group G as follows: $B \in G$ if and only if there exists $A \in \Gamma$ such that

$$\omega(B\zeta) = A\omega(\zeta). \tag{6.2}$$

It is evident that A is uniquely determined, and the mapping $A \to B$ is a surjective homomorphism whose kernel we denote by G_0.

The equation

$$\hat{\phi}(\zeta) = \phi(\omega(\zeta))\omega'(\zeta)^2 \tag{6.3}$$

sets up a one-one correspondence between quadratic differentials with respect to G and Γ. One verifies that $Q(\Gamma)$ corresponds to $Q(G)$ and $Q^*(\Gamma)$ to $Q^*(G)$. A similar correspondence between $B(\Gamma)$ and $B(G)$ is defined by

$$\hat{\nu}(\zeta) = \nu(\omega(\zeta))\overline{\omega'}(\zeta)/\omega'(\zeta). \tag{6.4}$$

The identity

$$\int_{\Omega/\Gamma} \phi\nu \, dx dy = \int_{D/G} \hat{\phi}\hat{\nu} \, d\xi d\eta \tag{6.5}$$

prevails. We conclude:

THEOREM 3. *The conjugate space of $Q(\Gamma)$ has a canonical identification with $Q^*(\Gamma)$.*

6.2. For the counterpart of Lemma 5 it is also true that it suffices to consider a component Ω_{ij} and the corresponding subgroup Γ_{ij}. To see this, let Θ_{ij} denote the theta-operator with respect to Γ_{ij}. If $\phi = \Theta_{ij}F$ in Ω_{ij} with $F \in A(\Omega_{ij})$, extend F to all of Ω by setting $F = 0$ outside of Θ_{ij}. At the same time ϕ is extended to all of Ω_i as a quadratic differential. It follows that $\phi = \Theta F$ in Ω_i, and this is what we wish to prove. It is therefore no restriction to assume again that Ω is an invariant region for Γ.

LEMMA 7. *Every $\phi \in Q(G)$ can be written as ΘF with $F \in A(\Omega)$.*

We use the same notations as before. By Lemma 6 it is possible to write $\phi = \Theta_0\phi_0$ with $\phi_0 \in Q(G_0)$. The relation

$$\phi_0(\zeta) = F(\omega(\zeta))\omega'(\zeta)^2 \tag{6.6}$$

defines F uniquely, and one verifies that $\Theta_0\phi_0(\zeta) = \Theta F(\omega(\zeta))\omega'(\zeta)^2$, and this makes $\phi = \Theta F$.

7. **Direct characterization of $N(\Gamma)$.** 7.1. In this section we shall derive some results which, as far as we are able to judge, cannot be obtained by reduction to the Fuchsian case. It will be convenient to assume, in this

connection, that the points 0, 1, ∞ belong to Σ. All invariantly formulated results are of course independent of this hypothesis.

The method we shall use is intimately connected with infinitesimal deformations of conformal structure. However, there will be no need to make this connection explicit.

Given $\nu \in B(\Gamma)$ we construct

$$(7.1) \qquad f(\zeta) = -\pi^{-1}\int_{\Omega} \nu(z)\left[\frac{1}{z-\zeta}-\frac{\zeta}{z-1}-\frac{1-\zeta}{z}\right]dxdy.$$

The integral converges because the rational function in brackets has a third order zero at ∞ and only simple poles at ζ, 0, and 1. It represents a continuous function in the finite plane. What is more, it satisfies a uniform Hölder-type condition

$$(7.2) \qquad |f(\zeta_1)-f(\zeta_2)| \leqq C\,|\zeta_1-\zeta_2|\log 1/|\zeta_1-\zeta_2|,$$

for instance for $|\zeta_1-\zeta_2| \leqq 1/2$. At ∞ it is of order $O(|\zeta|\log|\zeta|)$.

It is well known that f has locally square integrable distributional derivatives, and that

$$(7.3) \qquad f_{\bar{z}} = \tfrac{1}{2}(f_x - if_y) = \nu$$

almost everywhere (provided that we set $\nu = 0$ on Σ).

7.2. We shall prove:

Lemma 8. *$\nu \in N(\Gamma)$ if and only if $f = 0$ on Σ.*

The necessity is immediate. Indeed, for any fixed $\zeta \in \Sigma$ the bracketed expression in (7.1) is a holomorphic and integrable function F on Ω. Therefore, $\nu \in N(\Gamma)$ implies

$$\int_{\Omega} \nu F\,dxdy = \langle \Theta F, \nu\rangle = 0,$$

and hence $f(\zeta) = 0$.

To prove the sufficiency, let λ be a C^{∞} function which vanishes in a neighborhood of Σ. In addition we require that λ and $f\lambda_{\bar{z}}$ be bounded. Under these conditions, if $F \in A(\Omega)$ Stokes' formula yields, with the help of (7.3),

$$(7.4) \qquad \int_{\Omega} \lambda F\nu\,dxdy = \int_{\Omega} f_{\bar{z}}\lambda F\,dxdy = -\int_{\Omega} f\lambda_{\bar{z}} F\,dxdy.$$

If we can choose λ, depending on a parameter, in such a way that λ tends boundedly to 1 and $f\lambda_{\bar{z}}$ tends boundedly to 0, it will follow that $\langle \Theta F, \nu\rangle = 0$, and hence, by Lemma 7, that $\nu \in N(\Gamma)$.

We are assuming that $f=0$ on Σ. As before, $\delta(z)$ denotes the distance from z to Σ. We deduce from (7.2) that

$$|f| \leqq C\delta \log 1/\delta \tag{7.5}$$

for $\delta \leqq \delta_0$, say. We choose $\delta_0 \leqq e^{-1}$.

For the moment, let us ignore the condition that λ be of class C^∞. Given $\epsilon > 0$, let $\Lambda(t)$ be 0 for $0 \leqq t \leqq \epsilon$, 1 for $t \geqq 2\epsilon$, and linear between ϵ and 2ϵ. We shall set

$$\lambda(z) = \Lambda[(\log\log 1/\delta(z))^{-1}] \tag{7.6}$$

if $\delta < \delta_0$ and $\lambda(z) = 1$ if $\delta \geqq \delta_0$. For ϵ small enough, λ will be continuous and identically zero near Σ.

If $\delta(z)$ were differentiable, the obvious inequality $|\delta(z_1) - \delta(z_2)| \leqq |z_1 - z_2|$ would yield $|\delta_{\bar z}| \leqq 1$, and hence

$$|\lambda_{\bar z}| \leqq \epsilon^{-1}\delta^{-1}(\log 1/\delta)^{-1}(\log\log 1/\delta)^{-2}$$

in the part where λ is not constant. Since $(\log\log 1/\delta)^{-1} \leqq 2\epsilon$ for these values it follows that

$$|\lambda_{\bar z}| \leqq 4\epsilon\delta^{-1}(\log 1/\delta)^{-1},$$

and in view of (7.5) we obtain $|\lambda_{\bar z} f| \leqq 4C\epsilon$ throughout Ω. The desired conclusion follows.

The reasoning is completed by an obvious smoothing process, which we need not explain in detail. First, we replace δ in (7.6) by a smooth mean value δ^* obtained by convolution. In the relevant part of the plane δ^* will satisfy the same Lipschitz condition as δ. Secondly, Λ has to be replaced by a smooth function whose derivative is nowhere much greater than ϵ^{-1}. With these modifications the argument goes through exactly as before.

7.3. An alternative characterization of $N(\Gamma)$ is the following:

Lemma 9. *$\nu \in N(\Gamma)$ if and only if $f(Az) = f(z)A'(z)$ for all $A \in \Gamma$.*

The condition means that f/dz is invariant, in which case f is said to be an inverse differential.

The definition (7.1) of f shows that $f(0) = 0$. If the condition in the lemma is fulfilled it follows that $f(A0) = 0$ for all $A \in \Gamma$. But the points $A0$ are dense on Σ and f is continuous. Therefore f vanishes on Σ, and Lemma 8 shows that $\nu \in N(\Gamma)$.

Consider now the function

$$P_A(z) = f(Az)A'(z)^{-1} - f(z). \tag{7.7}$$

By use of (7.3) we obtain

$$(7.8)\qquad (P_A)_{\bar z} = \nu(Az)\overline{A'(z)}A'(z)^{-1} - \nu(z) = 0$$

almost everywhere; in particular, also on Λ. Furthermore, the derivatives of P_A are locally square integrable (except perhaps at $A^{-1}\infty$). It follows that P_A is analytic, and if it vanishes on Λ it must be identically zero. This proves the lemma.

A more detailed investigation of the singularities at ∞ and $A^{-1}\infty$ shows that P_A is always a quadratic polynomial. It is of some interest to note the formula

$$(7.9)\qquad P_{AB}(z) = P_A(Bz)B'(z)^{-1} + P_B(z).$$

7.4. The last lemma leads immediately to the principal result:

THEOREM 4. *If Γ is finitely generated, then $S = \Omega/\Gamma$ is of finite type.*

Indeed, all relations $f(Az) = f(z)A'(z)$ are consequences of the corresponding relations for the generators. On the other hand, each relation is equivalent to the vanishing of the coefficients of P_A, and thus to three linear conditions on ν. For a finitely generated group the space $B(\Gamma)/N(\Gamma)$ has thus finite dimension. By Lemma 3 and Theorem 3 this means that $Q^*(\Gamma)$ is finite dimensional, and hence the same is true of $Q(\Gamma)$. We apply Theorem 1 to conclude that S is of finite type.

8. Invariant regions. 8.1. It is possible to obtain more precise information by adding the hypothesis that one or several Ω_i are connected, and hence invariant under the full group Γ. Such groups have been called function groups.[4]

The fact that all invariant regions Ω_i have the same boundary Λ does not by itself preclude the existence of any number of such regions. However, the existence of nonelliptic fixpoints does impose a severe restriction. The following theorem was communicated to me by R. Accola.

THEOREM 5. *There are at most two invariant regions Ω_i, and if there are two they are simply connected.*

For the simple proof, which is purely topological, we refer to a forthcoming article by Accola.

8.2. If Γ is finitely generated we are able to prove a more precise result:

[4] L. Ford, Automorphic Functions (2nd ed., Chelsea, 1951), p. 64.

THEOREM 6. *Suppose that Ω_1 and Ω_2 are invariant regions for the finitely generated group Γ. Then $S = \Omega/\Gamma$ has only the components $S_1 = \Omega_1/\Gamma$ and $S_2 = \Omega_2/\Gamma$. Moreover, S_1 and S_2 are homeomorphic with isomorphic signatures.*

Since Γ is finitely generated we need not distinguish between $Q(\Gamma)$ and $Q^*(\Gamma)$. We shall denote by Q_i the subspace of quadratic differentials that vanish outside of Ω_i. The main part of the theorem is an easy consequence of

LEMMA 10. *If Ω_1 is connected, then* $\dim Q \leqq 2 \dim Q_1$.

Indeed, under the hypothesis of the theorem the lemma yields $\dim Q_1 + \dim Q_2 \leqq \dim Q \leqq 2 \dim Q_1$. Hence $\dim Q_2 \leqq \dim Q_1$, and by symmetry $\dim Q_1 = \dim Q_2$. Furthermore, $\dim Q = \dim Q_1 + \dim Q_2$, and this shows that there is no room for a third component. If we can show that S_1 and S_2 have the same signature it will follow, by the formula for the number of linearly independent quadratic differentials, that they have the same genus.

The points with finite signature arise from elliptic fixpoints, and it is easy to see that each elliptic transformation must have one fixpoint in Ω_1 and one in Ω_2 (the simplest way to see this is by conformal mapping of Ω_1 and Ω_2 on the unit disk). It is therefore evident that the points with finite signature can be matched in pairs, in such a way that corresponding points have the same signature.

The facts for points with infinite signature are quite similar. We know by Lemma 1 that any such point, whether on S_1 or S_2, determines a class of conjugate parabolic transformations. In the present situation the converse is also true. Let $A \in \Gamma$ be parabolic. We map Ω_1 conformally on the unit disk D and use the construction in 6.1 to determine an isomorphic mapping of Γ on a Fuchsian group G. The image B of A is necessarily parabolic. In the elementary case of a Fuchsian group it is quite evident that B corresponds to a point on S_1 with infinite signature (M. Heins has proved this for arbitrary Fuchsian groups).[8] Moreover, non-conjugate transformations correspond to different points. It follows that the points with infinite signature are indeed in one-one correspondence.

8.3. It remains to prove Lemma 10. Let us denote by Q'_1 the space of quadratic differentials which vanish on Ω_1. Given $\phi \in Q'_1$ we form $\nu = \bar{\phi}\rho^2$, which is a Beltrami differential, and construct

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[8] M. Heins, "On Fuchsoid groups that contain parabolic transformations" (*Contributions to function theory*), Tata Institute, Bombay, 1960.

$$S\phi(\zeta) = \int_{\Omega-\Omega_1} \nu(z)(z-\zeta)^{-4}\,dxdy \tag{8.1}$$

for $\zeta \in \Omega_1$. One verifies that $S\phi$ is a quadratic differential. Because ν is bounded the integral is majorized by

$$\int_{|z-\zeta|>\delta(\zeta)} |z-\zeta|^{-4}\,dxdy$$

which is a multiple of $\delta(\zeta)^{-2}$. Since Ω_1 is simply connected $\delta(\zeta)^{-1}$ is comparable with $\rho(\zeta)$ (we are assuming that $\infty \in \Sigma$). It follows that $S\phi \in Q_1$.

We shall show that the antilinear mapping $S: Q'_1 \to Q_1$ is one to one. Assuming that $0, 1, \infty \in \Sigma$ we construct, as in 7.1,

$$f(\zeta) = -\pi^{-1}\int_{\Omega-\Omega_1} \nu(z)\left[\frac{1}{z-\zeta} - \frac{\zeta}{z-1} - \frac{1-\zeta}{z}\right]dxdy. \tag{8.2}$$

If $S\phi = 0$ we see by comparison with (8.1) that $f'''(\zeta) = 0$ in Ω_1. Hence $f(\zeta)$ is a quadratic polynomial in Ω_1. Its behavior at ∞ shows that it is in fact a linear polynomial, and because it vanishes at 0 and 1 it reduces identically to zero. By continuity $f = 0$ on Σ, and we conclude by Lemma 8 that $\nu \in N(\Gamma)$. Since $\nu = \bar{\phi}\rho^2$ this gives $\phi = 0$, and we have proved that S is one to one.

We emphasize that the connectedness of Ω_1 is essential for the proof. If Ω_1 were not connected we could merely conclude that f is a quadratic polynomial in each component, but not necessarily the same in all components.

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