Chapter 1

Preliminaries

1.1 General notation

We shall denote by $\mathbb{N}$ the set of all natural numbers (we do not consider 0 as a natural number; thus $\mathbb{N}$ is the set of all positive integers). We shall also denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the set of all integer numbers, rational numbers, real numbers an complex numbers, respectively. The unit circle will be denoted by $\mathbb{S}^1$ and is defined to be the set $\{z \in \mathbb{C} : |z| = 1\}$.

We shall also use the symbols Cl, Int and Bd to denote the closure, the interior and the boundary of a set, respectively.

Let X be a topological space and let $\mathcal{X}$ be some class of maps from X to itself. We shall study the properties of the maps in $\mathcal{X}$ from the point of view of their iterates. The main objects we are interested in are the orbits of the points from X under the action of the maps from $\mathcal{X}$. To be precise we introduce some notation. For a map $f \in \mathcal{X}$ we use the symbol f^n to denote $f \circ f \circ \cdots \circ f$ (n times). Additionally we set $f^0 = \mathrm{id}$. Then, for a point $x \in X$ we define the *orbit* of x, denoted by $\mathrm{Orb}_f(x)$, as the set $\{f^n(x) : n = 0, 1, 2, \dots\}$. We will be interested usually in considering the restriction of a map to some finite orbit rather than the whole map. To this end we introduce the notion of a cycle. We say that (P, φ) is a *cycle* if $P \subset X$ is a finite set and φ is a cyclic permutation of P. Most of the time we shall identify a cycle (P, φ) with P itself, and say that P is a cycle. This kind of simplification is used in many cases in mathematics (e.g. when one defines a group). Usually it will be clear what φ is, since it will be a restriction of some given map in $\mathcal{X}$ to P. However, when doubts can arise, we shall use the precise notation.

The number of elements of P will be denoted by $|P|$ and will be called the *period* of (P, φ) (or just the period of P). Sometimes we shall call a cycle of period n an *n-cycle*.

If $f \in \mathcal{X}$ is a map and (P, φ) is a cycle we say that *P is a cycle of f*, or that *f has a cycle P* if $\varphi = f\big|_P$. To simplify the notation we shall use usually the symbol (P, f) to denote the cycle $(P, f\big|_P)$ of f.

We note that if a map $f \in \mathcal{X}$ has an n-cycle P and $x \in P$, then $P = \mathrm{Orb}_f(x)$. Such an orbit is often called a *periodic orbit of f of period n*. Also, every point of such a cycle will be called a *periodic point of f of period n* or an *n-periodic point of f*. A 1-periodic point (that is the unique element of a 1-cycle) will be called a *fixed point*. For a map $f \in \mathcal{X}$ we denote by $\mathrm{Per}(f)$ the set of all positive integers n such that f has an n-cycle.

The abstract notion of a cycle differs a little bit from the notion of a cycle of a map. One should look at a cycle in the context of the whole class $\mathcal{X}$ under consideration and think about all maps of $\mathcal{X}$ having this cycle.

Now we introduce a tool that will play an important role in the study of cycles.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set. We say that $G = (V, U)$ is an *oriented generalized graph* if $U \subset V \times V \times \mathbb{N}$ and for each i, j we have $\{k \in \mathbb{N} : (v_i, v_j, k) \in U\} = \{1, 2, \ldots, a_{ij}\}$ with $a_{ij} \geq 0$. The elements of V are the *vertices* of the oriented generalized graph G. Each element $(v_i, v_j, k) \in U$ will be called an *arrow* from v_i to v_j. Thus, the elements of U are the arrows of G. Clearly, for all i, j the number of arrows of G from v_i to v_j is a_{ij}. Let $G' = (V', U')$ be an oriented generalized graph. We say that G' is an *(oriented generalized)*

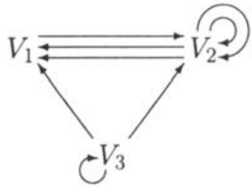

Figure 1.1.1: The oriented generalized graph (V, U).

subgraph of G if $V' \subset V$ and $U' \subset U$.

An arrow (v_i, v_j, k) will also be denoted by $v_i \xrightarrow{k} v_j$ or simply by $v_i \longrightarrow v_j$. This notation allows us to give a graphic representation of an oriented generalized graph.

Example 1.1.1. Let $V = \{v_1, v_2, v_3\}$ and let

$$U = \{(v_1, v_2, 1), (v_2, v_2, 1), (v_2, v_2, 2), (v_2, v_1, 1), (v_2, v_1, 2), (v_3, v_3, 1), (v_3, v_1, 1), (v_3, v_2, 1)\}.$$

Then the oriented generalized graph (V, U) is the one shown in Figure 1.1.1. □

Let $G = (V, U)$ and $G' = (V', U')$ be two oriented generalized graphs. We say that G and G' are *isomorphic* if there exists a bijective map $\phi : V \longrightarrow V'$ such that $U' = \{(\phi(v), \phi(w), k) : (v, w, k) \in U\}$.

An *oriented graph* (or simply a *graph*) is an oriented generalized graph in which the number of arrows from one vertex to another is at most one. An oriented generalized subgraph of a graph will be called a *subgraph*.

Let $\alpha = (u_1, u_2, \ldots, u_m)$ be a sequence of arrows of an oriented generalized graph G, where $u_j = (v_j, v'_j, k_j)$ for $j = 1, 2, \ldots, m$. We say that α is a *path of G of length m* if $v_{j+1} = v'_j$ for $j = 1, 2, \ldots, m-1$. The path α of G of length m is called a *loop of G of length m* if $v_1 = v'_m$. If α is a loop of G then $(u_2, u_3, \ldots, u_m, u_1)$ is also a loop of G. This loop will be called the *shift* of α and it will be denoted by $\Upsilon(\alpha)$. For the n-th iterate of Υ we shall use the symbol Υ^n. Notice that if α is a loop of length m then $\Upsilon^m(\alpha) = \alpha$. Let $\alpha = (u_1, u_2, \ldots, u_m)$ and $\beta = (w_1, w_2, \ldots, w_n)$ be two loops of an oriented generalized graph G, such that $u_1 = (v, v', k)$ and $w_1 = (v, v'', l)$. We define the *concatenation* of α and β as the loop $(u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_n)$ of G and we denote it by $\alpha\beta$. Lastly, if α is a loop of G, then we say that the loop $\alpha\alpha\ldots\alpha$ (n times) is a *n-repetition of* α. A loop of G will be called *simple* if it is not a n-repetition of any other loop of G with $n \geq 2$. Otherwise it will be called *repetitive*. For instance, $v_2 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_2$ is a simple loop of length 3 of the oriented generalized graph of the above example, while $v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_1$ is a loop of length 4 which is repetitive.

In the rest of this chapter we will restrict the choice of X to some particular one-dimensional topological spaces. Of course, in each case we have to specify the class $\mathcal{X}$. We shall use the following notation. By an *interval* we shall mean a nonempty bounded connected subset of $\mathbb{R}$. We note that a set consisting of a single point is an interval but the whole real line $\mathbb{R}$ is not. Given a set $A \subset \mathbb{R}$, by a *subinterval* of A we shall understand an interval contained in A. An interval will be called *degenerate* if it consists of a single point. Otherwise it will be called *proper*. From now on $\mathcal{I}$ will denote the set of all continuous maps from a closed proper interval I to itself. Similarly, $\mathcal{R}$ will denote the set of all continuous maps from $\mathbb{R}$ to itself. Often, a map from $\mathcal{I}$ will be called simply an *interval map*. Also, $\mathcal{S}$ will denote the set of all continuous maps from $\mathbb{S}^1$ to itself. A map from $\mathcal{S}$ will also be called a *circle map*. In what follows X will denote I, $\mathbb{R}$ or $\mathbb{S}^1$ and $\mathcal{X}$ will stand for $\mathcal{I}$, $\mathcal{R}$ or $\mathcal{S}$, respectively.

When we deal with maps from $\mathcal{I}$ or from $\mathcal{R}$ we often talk about subintervals of I or $\mathbb{R}$. For maps from $\mathcal{S}$ we have to replace them by arcs. Let e be the natural projection from $\mathbb{R}$ to $\mathbb{S}^1$. That is $e(x) = \exp(2\pi i x)$. By an *arc* we mean an image under e of an interval in $\mathbb{R}$ on which e is one-to-one, but we also should know the endpoints of this arc. If we choose the intervals $[0, 1)$, $(0, 1]$, $[1/2, 3/2)$ or $(1/2, 3/2]$, the image of each of them is the whole circle. However we should be able to distinguish between them. This explanation should be enough for the reader to understand what we mean by arc. However, for these readers who like very rigorous definitions, we can provide the following one. The *arc* is a 5-tuple (J, a, b, α, β) where $J \subset \mathbb{S}^1$, $a, b \in \mathbb{S}^1$ and $\alpha, \beta \in \{0, 1\}$ such that there exists an interval $K \subset \mathbb{R}$ with the property that $e(K) = J$; e is one-to-one on K; the image of the left endpoint of K is a and the image of the right one is b; K is closed from the left if $\alpha = 1$ and open from the left if $\alpha = 0$; and K is closed from the right if $\beta = 1$ and open from the right if $\beta = 0$. Of course, we shall always denote an arc (J, a, b, α, β) simply by J. To unify the notation, an arc of the circle usually will be called also an interval. Then we will also use the expressions: subinterval, open interval, closed interval, open from the left, open from the right, closed from the left and closed from the right in the obvious way.

Let $K, L \subset X$ be intervals and let $f \in \mathcal{X}$. We say that K *f-covers* L (denoted by $K \longrightarrow L$), if there exists a subinterval J of K such that $f(J) = L$. We say that K *f-covers L m times* if there exist m subintervals $J_1, J_2, \ldots, J_m$ of K with pairwise disjoint interiors such that $f(J_i) = L$ for $i = 1, 2, \ldots, m$.

1.2 Graphs, loops and cycles

We start this section by proving facts on f-covering which are elementary but nevertheless basic for the understanding and use of this notion.

Lemma 1.2.1. *Let $X = I$ or $\mathbb{R}$ and let $f \in \mathcal{X}$. Let K, L be proper closed subintervals of X. Then the following statements are equivalent.*

(a) K f-covers L,

(b) $f(K) \supset L$,

(c) there is a subinterval $J \subset K$ such that $f(\mathrm{Bd}(J)) = \mathrm{Bd}(L)$.

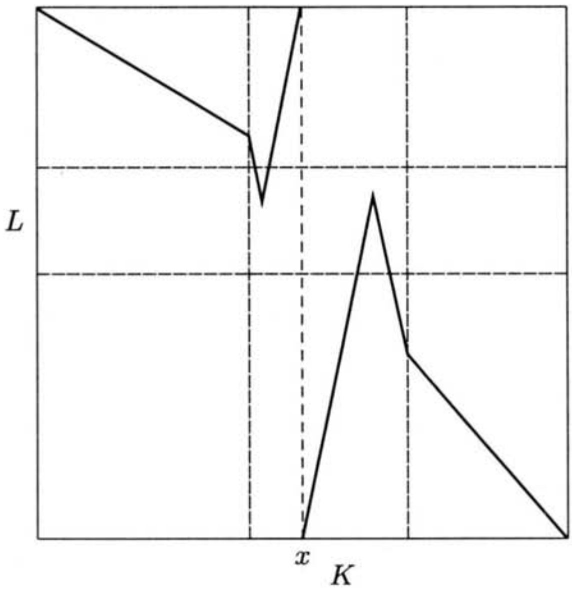

Figure 1.2.1: The graph of f in the 2-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$.

Proof. From the definition of f-covering it follows immediately that (a) implies (b). The fact that (b) implies (c) is obvious. To show that (c) implies (a), assume (c) and denote $J = [a, b]$. Then set

$$c = \sup\{x \in [a, b] : f(x) = f(a)\}$$

and

$$d = \inf\{x \in [c, b] : f(x) = f(b)\}.$$

Then clearly $f([c, d]) = L$, so K f-covers L. ■

We note that the above equivalent definitions of f-covering do not work for maps of $\mathcal{S}$. To see this, consider the map $f \in \mathcal{S}$ and the two intervals K and L given in Figure 1.2.1. Note that K does not f-cover L but $f(K) \supset L$ and there is a subinterval $J \subset K$ such that $f(\text{Bd}(J)) = \text{Bd}(L)$.

Corollary 1.2.2. *Let $X = I$ or $\mathbb{R}$ and let $f \in \mathcal{X}$. Let K, J and L be proper closed subintervals of X such that K f-covers L and $J \subset L$. Then K f-covers J.*

Remark 1.2.3. Although Lemma 1.2.1 does not hold when $X = \mathbb{S}^1$, Corollary 1.2.2 holds in this case. To see it, consider a map $f \in \mathcal{S}$ and proper closed intervals $K, J, L \subset \mathbb{S}^1$ such that K f-covers L and $J \subset L$. There exists a closed interval $M \subset K$ such that $f(M) = L$. Clearly, M is proper and it can be chosen closed. Next, there exist homeomorphisms $h_M : M \longrightarrow I$ and $h_L : L \longrightarrow I$. The map $g = h_L \circ f\big|_M \circ h_M^{-1}$ belongs to $\mathcal{I}$ and since $g(I) = h_L \circ f\big|_M (M) = I$, the interval I g-covers itself. By Corollary 1.2.2, I g-covers the interval $h_L(J)$, i.e. there exists a subinterval N of I such that $g(N) = h_L(J)$. But then $h_M^{-1}(N)$ is a subinterval of M and $f(h_M^{-1}(N)) = h_L^{-1} \circ g \circ h_M(h_M^{-1}(N)) = h_L^{-1} \circ g(N) = h_L^{-1}(h_L(J)) = J$. This proves that K f-covers J. □

Remark 1.2.4. Lemma 1.2.1, Corollary 1.2.2 and Remark 1.2.3 hold also for intervals which are not proper. In fact, in Lemma 1.2.1 conditions (a) and (b) are equivalent for arbitrary intervals K and L (not necessarily closed). We leave the proof of this fact to the reader. □

The following simple lemma shows that when a closed interval L f-covers itself then f has a fixed point in L.

Lemma 1.2.5. *Let $f \in \mathcal{X}$ and let $K \subset L$ be closed intervals of X such that $f(K) = L$. Then there exists $x \in K$ such that $f(x) = x$.*

Proof. If L is degenerate then the lemma holds trivially. Thus assume that L is proper. Then clearly K is also proper. Consider first the case when $X = I$ or $\mathbb{R}$. Then there are points $a, b \in K$ such that $f(a) = \min L$ and $f(b) = \max L$. We have $f(a) - a \leq 0$ and $f(b) - b \geq 0$, so by continuity of f there exists $x \in K$ such that $f(x) - x = 0$, i.e. $f(x) = x$.

Assume now that $X = \mathbb{S}^1$. Then there exists a homeomorphism $h : L \longrightarrow I$ and we can extend the map $h \circ f \circ (h\big|_K)^{-1} : h(K) \longrightarrow I$ to a continuous map $g : I \longrightarrow I$. We have

$$g(h(K)) = h \circ f \circ (h\big|_K)^{-1}(h(K)) = h \circ f(K) = h(L) = I$$

and by the part already proven there exists a point $y \in h(K)$ such that $g(y) = y$. Then for $x = h^{-1}(y)$ we have $f(x) = h^{-1} \circ g \circ h\big|_K (x) = h^{-1} \circ g(y) = h^{-1}(y) = x$. ■

Now we are going to show how the notions of f-covering and of an oriented generalized graph help in the study of cycles of maps of $\mathcal{X}$. We need several definitions.

If P is a non-empty subset of $X = \mathbb{R}$ or I, then $\langle P \rangle$ will denote the *closed convex hull* of P, i.e. the smallest closed connected subset of X containing P. This set is either an interval or a semi-line or the whole real line. To simplify the notation, instead of writing expressions like $\langle \{x, y\} \rangle$ or $\langle \{x, y, z\} \rangle$ we normally shall write simply $\langle x, y \rangle$ or $\langle x, y, z \rangle$, respectively.

Let $X = I$, $\mathbb{S}^1$ or $\mathbb{R}$ and let P be a subset of X. We assume that either P is finite or $X = \mathbb{R} = \langle P \rangle$ and P is discrete (that is, it is closed and has no accumulation points). If $X = \mathbb{S}^1$ then we assume also that P has at least two elements. Then the *quasipartition of X by the points of P* is the set $\mathcal{P}$ of all maximal proper closed intervals $J \subset X$ such that $\text{Int}(J) \cap P = \emptyset$. Note that $\bigcup_{J \in \mathcal{P}} J = X$ and if $K, J \in \mathcal{P}$, $K \neq J$ then $K \cap J$ consists of at most one point which belongs to P (that is, $\text{Int}(K) \cap \text{Int}(J) = \emptyset$). The elements of $\mathcal{P}$ contained in $\langle P \rangle$ (in the case of $X = \mathbb{S}^1$ all the elements of $\mathcal{P}$) will be called *P-basic intervals*.

Let $f \in \mathcal{X}$ and let P be a finite subset of X. An *f-graph of P* is an oriented generalized graph with the P-basic intervals as vertices and such that if K and J are P-basic intervals and K f-covers J m times, then there are m arrows from K to J (see Figure 1.2.2 for an example). Note that the f-graph of P is unique up to labelling of the P-basic intervals. Hence from now on we shall talk about *the* f-graph of P.

Now we are going to study the relation between cycles and loops in an f-graph. The next results show how f-graphs help in the study of cycles. Although Corollary 1.2.8 is simple, we shall prove it in three steps because we will sometimes use the intermediate results.

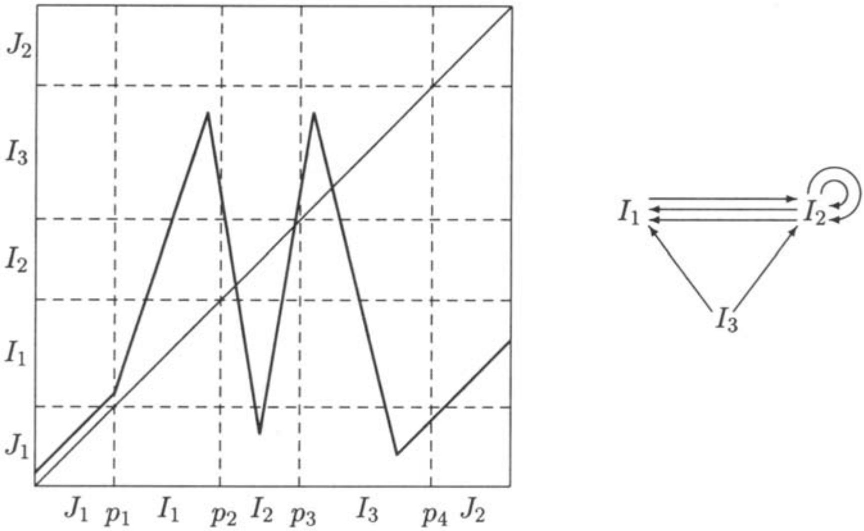

Figure 1.2.2: Here, $P = \{p_1, p_2, p_3, p_4\}$, $X = I = I_1 \cup I_2 \cup I_3 \cup J_1 \cup J_2$, and $\mathcal{P} = \{I_1, I_2, I_3, J_1, J_2\}$.

Lemma 1.2.6. *Let $I_0, I_1, \ldots, I_n \subset \mathbb{R}$ be closed intervals and, for $i = 0, 1, \ldots, n-1$, let $f_i : I_i \longrightarrow \mathbb{R}$ be continuous maps such that $f_i(I_i) \supset I_{i+1}$. Then there exist closed intervals $K_i \subset I_i$ $(i = 0, 1, \ldots, n)$ such that $K_n = I_n$ and $f_i(K_i) = K_{i+1}$ for $i = 0, 1, \ldots, n-1$.*

Proof. We shall use backward induction. Suppose that we have constructed $K_k \subset I_k$ for some $k > 0$. Then by Corollary 1.2.2 and Remark 1.2.3, I_{k-1} f_{k-1}-covers K_k and therefore there exists a closed interval $K_{k-1} \subset I_{k-1}$ such that $f_{k-1}(K_{k-1}) = K_k$. ∎

Lemma 1.2.7. *Let $I_0, I_1, \ldots, I_n \subset \mathbb{R}$ be closed intervals such that $I_n = I_0$ and, for $i = 0, 1, \ldots, n-1$, let $f_i : I_i \longrightarrow \mathbb{R}$ be continuous maps such that $f_i(I_i) \supset I_{i+1}$. Then there exist points $x_i \in I_i$ $(i = 0, 1, \ldots, n)$ such that $f_i(x_i) = x_{i+1}$ for $i = 0, 1, \ldots, n-1$ and $x_n = x_0$.*

Proof. Let K_i $(i = 0, 1, \ldots, n)$ be the intervals constructed in the previous lemma. Set

$$g = f_{n-1}\big|_{K_{n-1}} \circ f_{n-2}\big|_{K_{n-2}} \circ \cdots \circ f_1\big|_{K_1} \circ f_0\big|_{K_0}.$$

Then $K_0 \subset I_0$ and $g(K_0) = I_n = I_0$. By Lemma 1.2.5 there exists $x \in K_0$ such that $g(x) = x$. We define by induction $x_0 = x$ and $x_{i+1} = f_i(x_i)$ for $i = 0, 1, \ldots, n-1$. Then we have $x_i \in K_i \subset I_i$ for $i = 0, 1, \ldots, n-1$ and $x_n = x_0$. ∎

From the above lemma we get the following corollary.

Corollary 1.2.8. *Let $f \in \mathcal{X}$, let P be a finite subset of X and let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ be a loop in the f-graph of P. Then there exists a fixed point x of f^n, such that $f^i(x) \in I_i$ for $i = 0, 1, \ldots, n-1$.*

Proof. If $X = I$ or $\mathbb{R}$ then the assertion of the corollary is an immediate consequence of Lemma 1.2.7. If $X = \mathbb{S}^1$ then we set $I_n = I_0$ and apply Lemma 1.2.7 to the intervals $K_i \subset I_i$ such that $f(K_i) = I_{i+1}$ ($i = 0, 1, \ldots, n-1$), which exist by the definition of f-covering (since each I_i is homeomorphic to a real interval, we may treat I_i's and K_i's as intervals in $\mathbb{R}$). ∎

Unfortunately the converse of Corollary 1.2.8 is not true in general. Later we shall show that it holds for a particular class of maps (see Lemma 1.2.12). The following proposition illustrates how Corollary 1.2.8 can be used to prove the existence of cycles of various periods for interval maps.

Proposition 1.2.9. *Let $X = I$ or $\mathbb{R}$ and let $f \in \mathcal{X}$. Let K and J be two proper closed subintervals of X such that both K and J f-cover K and J. Assume that* $\text{Int}(K) \cap \text{Int}(J) = \emptyset$. *Then for each $n \in \mathbb{N}$ the map f has an n-cycle contained in $K \cup J$.*

Proof. We start by using Lemma 1.2.1 several times. Since f is continuous, $K \cup J \subset f(K)$ and $K \cup J \subset f(J)$ we have $L \subset f(K)$ and $L \subset f(J)$, where $L = \langle K \cup J \rangle$. Then there exist $K' \subset K$ and

$J' \subset J$ such that $f(\mathrm{Bd}(K')) = f(\mathrm{Bd}(J')) = \mathrm{Bd}(L)$. Thus both K' and J' f-cover K' and J'. Therefore for each $n \in \mathbb{N}$ there is a loop $K' \longrightarrow J' \longrightarrow J' \longrightarrow \cdots \longrightarrow J' \longrightarrow K'$ of length n in the f-graph of $\mathrm{Bd}(K') \cup \mathrm{Bd}(J')$ (if $n = 1$ then this is just $K' \longrightarrow K'$). By Corollary 1.2.8, f has a periodic point $x \in K'$ such that $f^i(x) \in J'$ for $i = 1, 2, \ldots, n-1$ and $f^n(x) = x$. If $x \notin J'$ then $f^i(x) \neq x$ for $i = 1, 2, \ldots, n-1$ and the period of x is n. In this case the proof is complete.

Suppose that $x \in J'$. Then x is a common endpoint of K' and J'. However, if $z \in \mathrm{Orb}_f(x) \cap (\mathrm{Bd}(K') \cup \mathrm{Bd}(J'))$ then $f(z) \in \mathrm{Bd}(L)$ and $f(z) \in \mathrm{Orb}_f(x) \subset K' \cup J'$, so $f(z)$ must be the common endpoint of L and $L' = K' \cup J'$. Since again $f(z) \in \mathrm{Orb}_f(x) \cap (\mathrm{Bd}(K') \cup \mathrm{Bd}(J'))$ we can use induction to show that $f^m(z) \in \mathrm{Bd}(L')$ for all $m \geq 1$. This gives a contradiction for $z = x$ and $m = n$, since $f^n(x) = x \notin \mathrm{Bd}(L')$. ■

Remark 1.2.10. The situation described in the statement of Proposition 1.2.9 will be called a "horseshoe" in Chapter 4 (see Page 204). We note that Proposition 1.2.9 does not hold in the case $X = \mathbb{S}^1$ because for the map $f \in \mathcal{S}$ given by $f(z) = z^{-2}$ the P-basic intervals I, J of the quasipartition of $\mathbb{S}^1$ by the set $\{1, -1\}$ satisfy the assumptions of this proposition but it is easy to check that f has no 2-cycles (see e.g. Lemma 3.4.3(a)). □

Now we are going to investigate more closely the relation between loops and cycles for maps form $\mathcal{I}$ or $\mathcal{R}$ and we shall give some results which allow us to compute the actual period of a cycle obtained from a loop in certain cases. So, in the rest of this section X denotes either I or $\mathbb{R}$ and $\mathcal{X}$ denotes $\mathcal{I}$ or $\mathcal{R}$, respectively. We introduce a new notion.

Let $f \in \mathcal{X}$ and let P be a finite subset of X. If Q is a cycle of f and $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{|Q|-1} \longrightarrow I_0$ is a loop of length $|Q|$ in the f-graph of P then we will say that α and Q are *associated* to each other if there exists $x \in Q$ such that $f^i(x) \in I_i$ for $i = 0, 1, 2, \ldots, |Q|-1$. We note that if α is a loop associated to a cycle Q, then all loops $\Upsilon^n(\alpha)$ are also associated to Q. We also note that if Q is associated to $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ then $Q \subset \bigcup_{i=0}^{n-1} I_i \subset \langle P \rangle$, so $\langle Q \rangle \subset \langle P \rangle$.

Lemma 1.2.11. *Let $f \in \mathcal{X}$ and let P be a cycle of f. Each simple loop from the f-graph of P has a cycle associated to it.*

Proof. Let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ be a simple loop in the f-graph of P. By Corollary 1.2.8 there exists $x \in X$ such that $f^i(x) \in I_i$ for $i = 0, 1, \dots, n-1$ and $f^n(x) = x$. Moreover, we may assume that the points $f^i(x)$ belong to the intervals K_i constructed in Lemma 1.2.6. Let k be the period of x and assume that $k \neq n$. Then $n = k \cdot m$ with $m > 1$. Clearly $f^i(x) = f^{i+k}(x) = \cdots = f^{i+k\cdot(m-1)}(x)$ for $i = 0, 1, \dots, k-1$. If $x \notin P$ then $\mathrm{Orb}_f(x) \subset \bigcup_{i=0}^{n-1} \mathrm{Int}(I_i)$. Therefore we get $I_i = I_{i+k} = \cdots = I_{i+k(m-1)}$ for all $i = 0, 1, \dots, k-1$. Hence, α is an m-repetition of the loop $I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_0$; a contradiction. Thus, if $x \notin P$ then $\mathrm{Orb}_f(x)$ is associated to α. Now we consider the case $x \in P$. We may assume that $x = \min P = \min I_0$. Let $K_0 \subset I_0$ be the interval constructed in Lemma 1.2.6. Then $f^i(x) \in f^i(K_0) \subset I_i$ for $i = 0, 1, \dots, n$. Since $x = f^k(x) \in f^k(K_0)$, it follows that $f^k(K_0) \subset I_0$ and hence $I_k = I_0$. Both sets K_0 and $f^k(K_0)$ are contained in I_0 and contain $\min I_0$. Therefore the interval $K_1 = K_0 \cap f^k(K_0)$ is proper. Clearly $f^i(K_1) \subset f^i(K_0) \subset I_i$ and $f^i(K_1) \subset f^{k+i}(K_0) \subset I_{k+i}$. Hence we get $I_i = I_{k+i}$ for $i = 0, 1, \dots, k-1$ and $f^k(K_1) \subset f^k(K_0) \subset I_0$. Since $f^k(K_1) \subset f^{2k}(K_0) \subset I_{2k}$ we also get $I_0 = I_{2k}$. Repeating this process we get that the loop α is not simple; a contradiction. This completes the proof. ∎

The next result is a kind of converse of Corollary 1.2.8. To state it we introduce the following notation.

Let f be a map from some space Y into itself and let P be a subset of Y. We say that a set $A \subset Y$ is *f-invariant* (or simply *invariant* if no confusion is possible) if $f(A) \subset A$. Of course every cycle of f is f-invariant. We also note that if A is f-invariant and P is a cycle of f, then either $P \subset A$ or $P \cap A = \emptyset$.

Let $f \in \mathcal{X}$ and let P be a finite subset of X. We say that f is *P-monotone* if it is constant on each of the two connected components of $X \setminus \langle P \rangle$ and f is monotone on each P-basic interval. If additionally f is affine on each P-basic interval then we shall call f *P-linear* (sometimes we will call it a *$(P, f|_P)$-linear*). We get the graph of a P-linear map from the graph of $f|_P$ by joining the points by segments of straight lines and drawing horizontal segments outside $\langle P \rangle$ (if $\langle P \rangle$ is a proper subset of I). We note that if f is P-linear then $f(p_i) \neq f(p_{i+1})$ for $i = 1, 2, \ldots, n-1$.

Lemma 1.2.12. *Let $f \in \mathcal{X}$ and let P be a finite invariant subset of X. Assume that f is P-monotone and let Q be a cycle of f such that $Q \cap P = \emptyset$. Then there exists a unique loop (modulo shifts) in the f-graph of P, associated to Q. If additionally P is a cycle then there is also a unique loop (modulo shifts) associated to P.*

Proof. Let $x \in Q$. We have $f^i(x) \in I_i$ for some P-basic interval I_i for all $i = 0, 1, 2, \ldots, n-1$, where $n = |Q|$. Since $Q \cap P = \emptyset$, we have $f^i(x) \notin P$ and hence $f^i(x) \in \mathrm{Int}(I_i)$ (of course for every i there is a unique I_i with this property). Therefore $f(\mathrm{Int}(I_i)) \cap \mathrm{Int}(I_{i+1}) \neq \emptyset$ for $i = 0, 1, \ldots, n-2$ and $f(\mathrm{Int}(I_{n-1})) \cap \mathrm{Int}(I_0) \neq \emptyset$. Since f is P-monotone and P is f-invariant, I_i f-covers I_{i+1} for all $i = 0, 1, 2, \ldots, n-2$ and I_{n-1} f-covers I_0. Then $I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ is a loop in the f-graph of P.

Now assume that P is a cycle. Set $x_0 = \min P$. Then there exists a unique P-basic interval I_0 such that $x_0 \in I_0$. Since f is P-monotone, $f(I_0)$ contains a unique P-basic interval I_1 such that $x_1 = f(x_0) \in I_1$. By Corollary 1.2.2 I_0 f-covers I_1. We continue this process and when we go back to $x_{|P|} = x_0$ then we get $I_{|P|} = I_0$ because it is the unique P-basic interval containing x_0. In such a way we obtain a unique loop $I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{|P|-1} \longrightarrow I_0$ associated to P. ■

Now we shall consider the notion of a graph associated to a cycle and we shall study its relation with the notion of an f-graph. We introduce some more notation.

Let (P, φ) be a cycle with $P = \{p_1, p_2, \dots, p_n\}$. Usually we will consider two labellings of the elements of P. We will say that the cycle P has the *spatial labelling* if $p_i < p_j$ if and only if $i < j$ and we will say that it has a *temporal labelling* if $p_i = \varphi^{i-1}(p_1)$ for $i = 2, 3, \dots, n$. We note that the above definition of temporal labelling is equivalent to $p_i = \varphi(p_{i-1})$ for $i = 2, 3, \dots, n$ and $p_1 = \varphi(p_n)$. Of course for a cycle (P, φ) we have $|P|$ different temporal labellings depending on which element of P we denote by p_1.

Let (P, φ) be a cycle with $P = \{p_1, p_2, \dots, p_n\}$ and assume that P has the spatial labelling. We define the *(P, φ)-graph* (or just the *P-graph*) as the graph with the P-basic intervals as vertices and such that there is an arrow from $[p_i, p_{i+1}]$ to $[p_j, p_{j+1}]$ if and only if $[p_j, p_{j+1}] \subset \langle \varphi(p_i), \varphi(p_{i+1}) \rangle$. We note that the notion of P-graph is fairly different from the notion of f-graph where f is a map from $\mathcal{X}$. In fact the notion of P-graph only depends on the cycle (P, φ) and does not depend on any map from $\mathcal{X}$. However, if $f \in \mathcal{X}$ and P is a cycle of f in some cases we will be interested in the (P, f)-graph. Clearly the (P, f)-graph involves only $f|_P$ but not the whole map f. From all we have said the following simple lemma follows.

Lemma 1.2.13. *Assume that a map $f \in \mathcal{X}$ has a cycle (P, φ). Then the (P, φ)-graph is a subgraph of the f-graph of P.*

Proof. Suppose that the cycle $P = \{p_1, p_2, \dots, p_n\}$ has the spatial labelling. Clearly, $\langle f(p_i), f(p_{i+1}) \rangle \subset f([p_i, p_{i+1}])$ for $i = 1, 2, \dots, n-1$. This completes the proof. ∎

To illustrate the above lemma, in Figure 1.2.3 we show a map $f \in \mathcal{X}$ having a cycle P, the f-graph of P and the (P, f)-graph. This figure shows that in general the (P, f)-graph differs from the f-graph of P. In the next result we show that for P-monotone maps this is not the case.

Lemma 1.2.14. *Let P be a cycle and let $f \in \mathcal{X}$ be P-monotone. Then the (P, f)-graph is equal to the whole $f|_{\langle P \rangle}$-graph of P.*

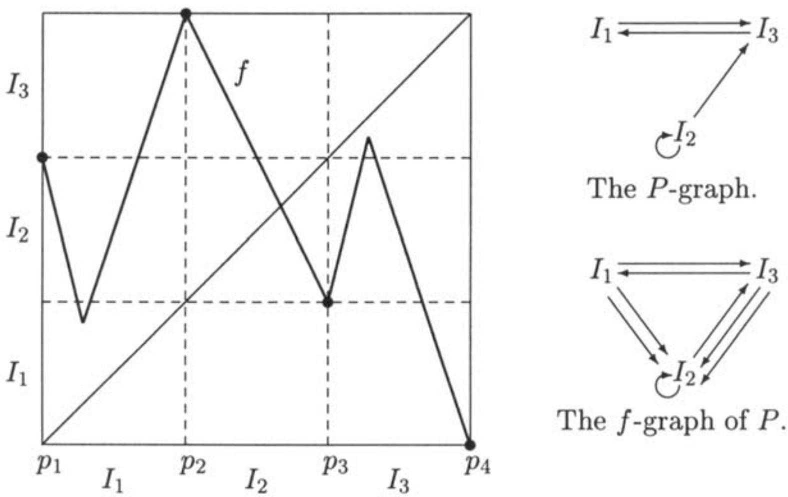

Figure 1.2.3: $P = \{p_1, p_2, p_3, p_4\}$, $I_1 = [p_1, p_2]$, $I_2 = [p_2, p_3]$ and $I_3 = [p_3, p_4]$.

Proof. Assume that $P = \{p_1, p_2, \ldots, p_n\}$ has the spatial labelling. Since f is P-monotone, it follows that $\langle f(p_i), f(p_{i+1})\rangle = f([p_i, p_{i+1}])$ for $i = 1, 2, \ldots, n-1$. This completes the proof. ■

♣ Sometimes an f-graph of P or a (P, φ)-graph is called a *Markov graph*. ♣

Historical remarks

The notion of f-covering was introduced by Block [73]. In his paper, Lemma 1.2.2, Corollary 1.2.8 and Remark 1.2.3 were proved and used to show the existence of cycles of certain periods. The idea of using the notion of f-graph of P to study the cycles of a map is due to Straffin [378], Block, Guckenheimer, Misiurewicz and Young [84], Ho and Morris [224] and Burkart [127].