Chapter 1

Introduction

1.1. Invariant Structures Everywhere

What do we mean when we say "invariant structures" or "invariant set"? According to the encyclopedic dictionary the noun "invariant" originates from the latin "invarians" (changeless) and means a value (a quantity) which is left unchanged by certain transformations. If this property is displayed not by a single value, but a set of values (or points), then such a set is referred to as invariant set. Mathematicians give more rigorous definitions using special mathematical concepts. One of such definitions will be given below, when we discuss dynamical systems. A simple example of an invariant set is provided by the collection of figures with a fixed area on the plane under an isometry (or any area-preserving transformation). A square with all four corners positioned on the coordinate axes in the plane (thus centered at the origin) is an invariant set under the rotation by 90^0. This type of invariance underlies the concept of symmetry. Another example of an invariant set is given by the orbit of the Earth under the transformations in the space of the Solar system induced by the planetary motions in time.

The Introduction and Part I of the book are intended for a wide audience and presuppose no special preliminary background. The only exception may be the notion of "dynamical system" which is closely related to the concept of "differential equations". The reader who is not familiar with (or interested in) differential equations may limit his or her reading by our consideration of discrete dynamical systems (i.e. maps) and experiment with the fractals generated by them. In this part of the book we mainly focus on the WInSet computer program and the results obtained by using it. The program not only enables the user to observe the invariant sets of dynamical systems (in particular, the fractals), but also allows one to introduce his own dynamical system and investigate its behavior and the fractals it generates.

Its sofisticated interface leads the user into the wonderful world of fractals, a world full of fascinating colors and harmony.

Part II is devoted to the description of invariant sets and demands from the reader the mathematical background in the amount of the first three years of college math. After having read it you will become familiar with the mathematical nature of the phenomena you observe on the computer screen when using, for example, WInSet software. Behind them are the frontline problems of the nonlinear dynamics, such as the nonlinear resonance, self-oscillations, irregular invariant sets (strange attractors, fractals etc.)

Shortly speaking, the main topic of the book is the visualization of invariant sets, and the demonstration of their beauty and somewhat mysterious inner harmony. We hope that this visual splendor will provide the inspiration (so important in the progress of sciences) for those who have a scientific interest in the nonlinear phenomena, as well as for those who are just curious about all this marvelous pictures.

1.1.1. Resonance Structures in Celestial Mechanics

Celestial mechanics is one of the most fascinating sciences, since it deals with the fundamental laws which govern the universe. It may even be considered the cradle of the calculus and the theory of differential equations and dynamical systems (for it is the differential equations that describe the motions of celestial bodies). In celestial mechanics the equations are usually assumed to be conservative, i.e. they preserve a certain quantity (the full energy of the system, for example). This is exactly the invariance property. The conservative systems are normally presented in a special form - as a so-called Hamiltonian system. In the modern mechanics and mathematics there is a field called Hamiltonian mechanics which deals with the general properties of this type of systems. The Hamiltonian systems generate resonance structures of astounding beauty (see Chapter 4).

Although a significant progress has been achieved in the study of celestial mechanics, one crucial question remains open. It is the question of the stability of Solar system. Whether the configuration of Solar system will remain stable over an infinite interval of time is not known. So it is unclear whether, for instance, Earth will always travel along an elliptic orbit close to the current one, or it will undergo a significant deformation and our planet will fall on the Sun or, on the contrary, will leave the Solar system altogether. The same question could be posed about the satellites of the planets. The question has been studied for more than two hundred years and is still unanswered. So, it is not known whether the current configuration of the Solar system is a result of a long evolutionary process or it "was born like this". The evolutionary model seems more plausible. In favor of it speaks the analysis of quasi-conservative systems and, most of all, the resonance structures in such systems.

The phenomena of "capture in a resonance" and "synchronization" can explain the "resonance" structure of the Solar system.

The **macroproblems** of celestial mechanics are similar, in their nature, to the **microproblems** on the structure of atom. Both are the problems on the motion of a "particle" in a central field. In the former problem the "particle" is a planet, in the latter — the electrons.

The resonance structures are invariant ones. Further in the book (Chapters 4-6) we shall discuss these structures in more detail by considering some simplified models. Right now let us look at the resonance equations in the solar system. Currently the scientists know with good precision the mean daily motions of the planets ω_j, $j = 1, ..., 9$ (expressed in degrees). So, one can check if the resonance equations are satisfied.

A resonance is said to occur if the equality

$$\sum_{j=1}^{n} k_j \omega_j = 0,$$

holds, where k_j are integer numbers, and n is the number of frequencies. If all planets are taken into considerations, then $n = 9$.

The resonance condition is the condition of commensurability of the frequencies. The number $||k|| = \sum_{j=1}^{n} |k_j|$ is called the order of the resonance. The main perturbing body in Solar system is Jupiter whose mass is many times larger than the masses of the other planets. So, let us consider the lowest resonances in the three body problem "Sun-Jupiter-planet". The lowest order of resonance $||k|| = 3206$ observed for the system Sun-Jupiter-Neptune is very large [29]. It turns out that the higher is the order of the resonance the less it is manifested in the motion of the bodies. Let us then consider not the exact resonances, but rather those close to the exact ones, i.e. instead of the resonance equality consider the condition in the form of inequality:

$$k_1 \omega_1 + k_2 \omega_{\text{пл}} \leq \varepsilon,$$

where $\varepsilon > 0$ is small enough, ω_1 is the mean daily motion of the Jupiter, and ω_{pl} is the mean daily motion of the planet. If, for instance, $\varepsilon = 5''$ (5 seconds), then the lowest order of resonance in the system Sun-Jupiter-Saturn is 7 ($k_1 = 2, k_{pl} = 5$), and the highest one is 649 ($k_1 = 617, k_{pl} = 32$) observed in the system Sun-Jupiter-Venus [29]. For the inner (with respect to Jupiter) planets increasing the "aperture" ε significantly lowers the maximal order of resonance.

Resonance relations are also observed in the satellite systems, as well as in planet-satellite type of systems. Laplace noticed the remarkable triple-frequency resonance in the satellite system of Jupiter formed by the mean motions of Jo, Europe and Ganimedes: $\nu_1 - 3\nu_2 + 2\nu_3 \approx -0.0003°$ (where ν_1, ν_2, ν_3 are the mean motions of Jo,

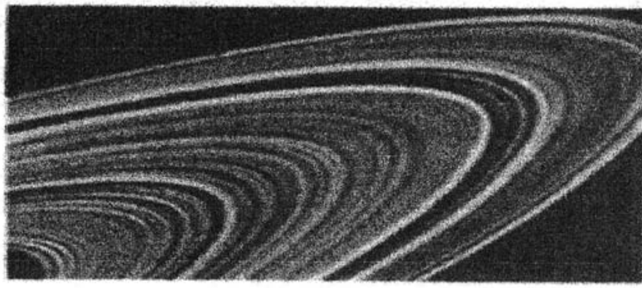

Fig. 1.1. The structure of the rings of Saturn (fragment).

Europe and Ganimedes respectively. The system Earth-Moon has the resonance with $k_1 = k_2 = 1$.

The satellite system and the rings of Saturn is a unique formation. Fig 1.1 shows a fragment of Saturn's system of rings obtained by Voyager-2 in August 1981 (reproduced from [93]). The structure of the rings looks somewhat like a fractal. Such invariant structures will be discussed in Section 1.1.3. and Chapter 7.

Another remarkable formation in the solar system is the belt of small planets (asteroids) which consists of more than 1800 objects. The statistical distribution of asteroids according to their mean motion is extremely irregular. Amazingly, some of its maxima and minima occur at the resonance values of the mean motion in system Jupiter-asteroid [29].

1.1.2. Cellular, Spiral, Vortex and Crystal Structures

Let us turn to more "earth-bound" sciences. In everyday life we meet (usually unaware of it) myriads of fascinating invariant structures of very diverse nature. In this section we briefly discuss some of those beauties. A good reading on various cases of formogenesis (formation of structures) in physics, chemistry, biology etc. is the recent book by Rabinovich and Ezersky [84].

Almost everyone heard that the structure of a honeycomb is hexagonal (the hexagons fit together, side to side, to form a "tiling" of the plane). A structural regularity can be observed in the configurations of flowers, seeds, leaves. The leaves of a shoot of a plant and the seeds in the sunflower display a regular *spiral* structure. There is a tendency towards a spiral in Nature. The botanists call it *philotaxis.*

Perhaps the most striking manifestation of Nature's romance with the spiral is the beautiful coach shell Nautilus pompilius [1].

Following Henry Weyl [104] we now describe one mathematical interpretation of the harmony of all these phenomena of Nature. One of the most common motions of a body in the three-dimensional space is the combination of a rotation around an axis with a translation along that axis. The trajectory of every point (outside the axis) under this motion yields a helical curve. Let us present the angle of the rotation in such a motion in the form $\frac{p}{q}360^0$, where p, q are relatively small integers. It has been observed that for the spiral describing the distribution of leaves around a central stem in plants the fractions $\frac{p}{q}$ are formed by the consequent Fibbonacci numbers:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots,$$

which converge to the irrational number $(\sqrt{5}-1)/2$, which is the famous "golden mean".

It is worth noting that the above expression for the rotation angle in the spiral motion is similar to the resonance frequency in a two-mode system (e.g. in celestial mechanics), where p and q represent the order and the type of the resonance, respectively. This similarity can be interpreted as some kind of "resonance" in the world of plants.

Another example coming from physics is the geometry of convective motions. These motions give rise to the convective cells which could assume the shape of one-dimensional rollers or form a quadratic or hexagonal lattice (Benar cells). Let us consider the convection in a heated layer of liquid — the so-called Rayleigh-Benar convection.

Following Berge [14] we consider the horizontal layer of liquid (silicon oil) bounded from above and from below by rigid heat-conductive plates. If the Rayleigh number Ra (proportionate to the difference of temperatures in the vertical direction) exceeds certain critical value Ra_*, then sets in a convective motion taking the shape of rotating rollers. Notice, that the velocity field everywhere has zero projection onto the axis of the rollers (see Figs. 1.2, 1.3(a)). When Ra increases even further the structure persists until Ra exceeds another fixed value $Ra_{**} > Ra_*$. The third component of the velocity field becomes non-zero, and the secondary set of rollers appears, with axes perpendicular to those of the primary rollers (Fig. 1.3(b)). In Fig.1.3(b) one cane notice a similarity with a two-dimensional crystal. Values Ra_*, Ra_{**} are called bifurcational values.

Further increase of the Rayleigh number leads to the appearance of local contraction near the boundary (pinch effect) when $Ra = 20Ra_*$. If we now stop increasing Ra

[1] This shell is held by the dancing Shiva of the Hindu myth as one of the instruments through which he initiates creation (translator's note.)

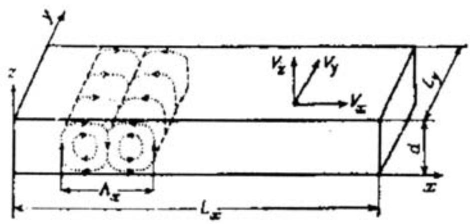

Fig. 1.2. Scheme of the convective motions of liquid in a rectangular configuration.

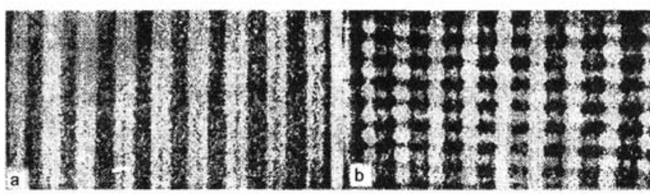

Fig. 1.3. Two-dimensional (a) and three-dimensional (b) structures.

and watch how the system evolves in time when $Ra \approx 20Ra_*$ if fixed (Fig. 1.4). Soon after the contraction begins there appears a isolated defect, around which the regular structure quickly disintegrates. This disintegration is followed by a metamorphosis which involves creation of a polygonal, cellular convective structure (Fig. 1.4 (d)). After 15 hours the entire structure disintegrates (melts) which is seen in Fig. 1.4 (f). Thereafter, the picture is changing continuously in a random way and the turbulence appears.

If we replaced the rectangular reservoir by a cylindrical one with diameter $D = 20d$, where d is the height of the cylinder, we would have got new invariant structures, some of which are shown in Fig. 1.5.

The so-called vortex structures are well known in hydrodynamics. They appear when a liquid flows round certain bodies. Analogous structures are known in aerodynamics. Let us at look at some experiments illustrating the vortex structures.

The following experiment was designed by Gak [16] (our discussion of its results below follows [20]). The experiment is based on the use of a magneto-hydrodynamic drive which allows to create a spatially periodic electromagnetic field in a thin layer of a weekly conducting liquid (electrolyte). Initially, the experiments of this type were conducted in a cuvette of dimensions 24 cm × 12 cm, 2–3 mm deep and with spatial period of 4.4 cm. By changing the density of the electric current between the

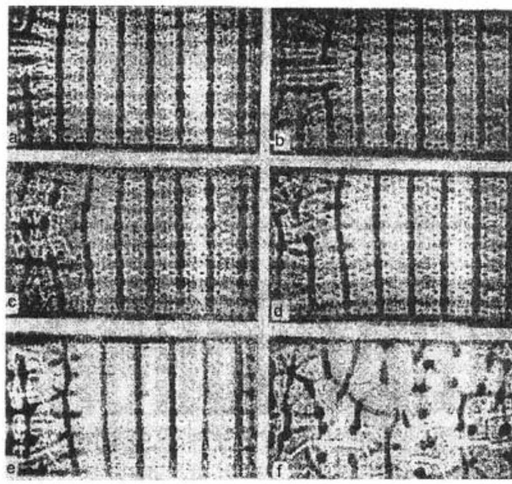

Fig. 1.4. Stages of "melting" of a three-dimensional structure at $Ra \approx 20Ra_8$.

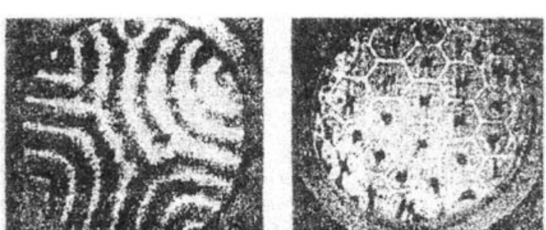

Fig. 1.5. Convective structures in a cylindrical reservoir.

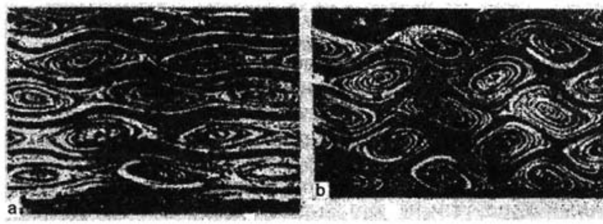

Fig. 1.6. Vortex structures for $Re/Re_{cr} = 1.1$(a) and $Re/Re_{cr} = 1.25$ (b).

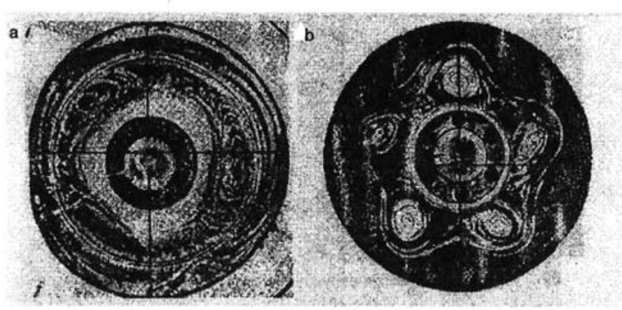

Fig. 1.7. Vortex structures in a round cuvette.

two electrodes immersed in the layer of liquid, one can create the flow with various Reinolds numbers (i.e. with different velocities of the liquid's movement). For some critical value of the Reinolds number (Re_{cr}) there appear "secondary" currents. In Fig. 1.6 the reader can see the picture of the secondary currents for $Re/Re_{cr} = 1.1$ (a) and $Re/Re_{cr} = 1.25$ (b).

Obukhov et al who experimented with a round cuvette also observed the secondary vortex currents [27, 20] (see Fig. 1.7).

The picture of the secondary currents presented in Fig. 1.7 (b) looks similar to the chain of vortices observed over Antarctica. The vortex structures shown in Figs. 1.6, 1.7 resemble the resonance structures in systems with 3/2 degrees of freedom (see Chapter 4).

Now let us talk about crystals. The most popular example of a crystal is a snowflake which has a hexagonal shape. In general, the symmetry in crystals can only have order 2, 3, 4 and 6 (see [104]). One may ask why the snowflakes cannot have any other regular form, a pentagon, for example? Do the laws of Nature forbid certain geometrical shapes? Such question intrigued people since great antiquity. In modern crystallography the use of geometric ideas is especially manifest. It is based on the theory of periodical structures that fill the space or the plane. The problem of realization of possible types of crystallic lattices can be reduced to the problem of constructing certain mosaics (tilings) of space or the plane with a prescribed type of symmetry. The literature on the theory of tilings is vast. The concept of "regularity", which has formerly been related to a certain kind of symmetry, is now being reconsidered. This rethinking is motivated by some recent discoveries in nonlinear dynamics. One and the same system can have, depending on the values of its parameters, different invariant regular structures. These structures may, however, be not only of regular, but also of irregular nature. In particular, this broader view of regularity is reflected in the concept of symmetry of "quasi-crystallic" type introduced by Zaslavsky et al. [109]. More details on this can be found in Chapters 4, 5.

1.1.3. Fractals

The term "fractal" was first introduced by Mandelbrot in the original 1975 version of his famous treatise [49]. Mandelbrot derived the word from the Latin "fractus" meaning "broken". He defined a fractal to be a set with Hausdorff dimension strictly greater than its topological dimension [2]. Although the definition is elegant and catches an important characteristic of fractals, it proved to be unsatisfactory, since it leaves out many sets that should definitely be regarded as fractals. A different definition proposed by Lauwerier [43] focuses on another important property of fractals — self-similarity. According to [43] *a set is a fractal if every portion of it is a scaled down copy of the entire set.* This definition seems satisfactory for an informal discussion, but, obviously, look precision and rigour, and therefore cannot be generally accepted. The search for a perfect definition continues...

The fractals that are used to be regarded as mathematical oddities now enjoy a lot of attention. Fractals are often referred to as "computer art", "esthetic chaos", etc.

The idea of the fractal structure can be illustrated by the binary tree, whose stem bifurcates into two branches, each of which bifurcates into two smaller branches, and so on ad infinitum. Every branch of this tree can be viewed as a stem carrying a binary tree which is a scaled down copy of the entire (primary) tree. One of the

[2]The topological dimension is always a non-negative integer. Any finite or countable set has topological dimension 0. Then, inductively, the topological dimension of the set equals $n+1$, if each point has a small neighborhood with boundary of dimension n. Thus, a smooth curve has topological dimension 1, a smooth surface — 2, a regular three-dimensional body — 3, etc. (translator's note)

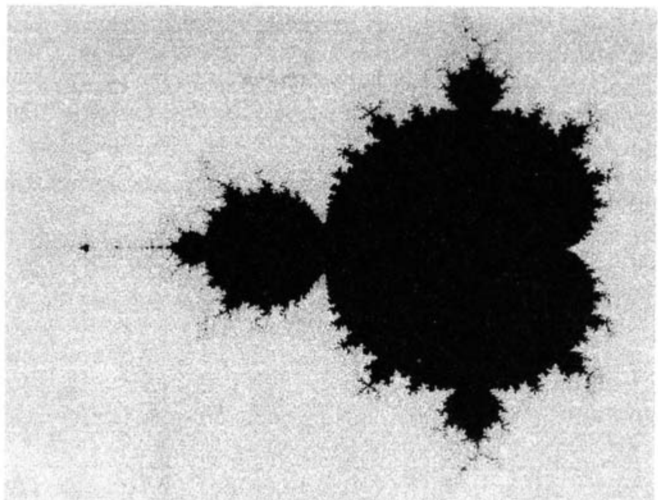

Fig. 1.8. Mandelbrot fractal

oldest and most fundamental examples of a fractal is the Cantor set. Other classical examples include the Sierpinski gasket, the von Koch curve and the graph of the Weierstrass function. In Section 7.3 we shall consider more "modern" fractals and show what they look like on a computer screen, using the WInSet software (which is enclosed with the book). One of such fractals, the so-called Mandelbrot set, is shown in Fig. 1.8.

If you have already installed the WInSet program, you can observe this (and other) fractals on your screen, you can also zoom a portion of the fractal and really see its self-similarity. You can also check (convince yourself) that the fractal is invariant with respect to the transformation by which it is generated. The fractal structure is especially transparent in color, and the algorithm of color control is easy to use (see Chapter 3 and [103]).

One of the earliest examples of fractals obtained as invariant sets were described in the seminal 1918 memoir of Gaston Julia. However, no pictures had been given in the memoir and the work was neglected for half a century. Computer graphics has mad visible the objects which could not be pictured in the time when Julia's work appeared. The images that were finally visualized have surpassed all expectations.

The concept of a fractal has proved useful in many applications [3]. In particular, it can broaden the notion of similarity by allowing random perturbations in the similarity maps. If random perturbations are introduced into a mathematically regular dendritic fractal, the result can come out looking like a real tree, coral or fungus.

Rather remarkable is the case of a jellyfish which possesses a symmetrical shape and, besides, is a kind of a living fractal structure! According to [104] "...as soon as it is born, a medusa starts to pulsate, a bell starts to "sound". Gemmae, miniature copies of the parent organism, appear on the tentacles, the stem and, sometimes on the fringe". It has been observed, for instance, that a small medusa Obelia gemmates and assumes its final shape with incredible speed, which makes us suspect a fractal structure.

The modern studies of fractals are associated with names of Poincaré, Julia and Mandelbrot and are mainly carried out in the context of the theory of dynamical systems. A dynamical system is, basically, a model of a material object which moves in a power field, like the planets of the solar system or elementary particles in an accelerator. The coexistence of populations of two different species is described by the dynamical system of Volterra. Generally speaking, any process or transformation can be thought of as a dynamical system. From the mathematical point of view a dynamical system is a system of differential or difference equations. Hamiltonian systems are an important class of dynamical systems. The Julia and Mandelbrot fractals are generated by a system of two discrete (difference) equations, or, equivalently, one complex equation. The rest of the book deals with dynamical systems. So, let us now define some important terms and concepts related to dynamical systems.

1.2. Dynamical Systems

Processes evolving in time can be described by some equations, which, as a rule, are differential ones. The set of such equations describing the process in question will be referred to as a **dynamical system**. Giving a more rigorous definition of a dynamical system requires some knowledge of calculus, which we henceforth assume the reader to have.

[3] You can actually use fractals to successfully predict the prices of stocks, commodities and currency (see [79] for details).

Consider the system of ordinary differential equations (ODE)

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2, ..., x_n)\\ \frac{dx_2}{dt} &= f_2(x_1, x_2, ..., x_n)\\ &\cdots\cdots\cdots\cdots\\ \frac{dx_n}{dt} &= f_n(x_1, x_2, ..., x_n),\end{aligned} \tag{1.1}$$

which can be presented in a more compact vector form

$$\frac{dx}{dt} = F(x), \quad x = (x_1, ..., x_n); \quad F = (f_1, ..., f_n). \tag{1.2}$$

System (1.1) or (1.2) is called a dynamical system and the number n is called the order or the dimension of the system.

The notion of dynamical system can be extended to include the systems of differential equations in which the right-hand sides explicitly depend on time:

$$\frac{dx}{dt} = F(x, t), \quad x = (x_1, ..., x_n); \quad F = (f_1, ..., f_n). \tag{1.3}$$

Such systems are said to be non-autonomous (as opposed to the systems of the form (1.2) which are said to be autonomous). We shall be dealing with systems (1.3) where function $F(x,t)$ is periodic in t. The study of such systems can be reduced to the study of autonomous system of dimension $n+1$. Thus, we shall only discuss systems of the form (1.2) here.

In mathematics there exists a more abstract and, hence, more general concept of a dynamical system which, basically, defines a dynamical system to be a *transformation* of some space into itself (see e.g. [2]).

The n-dimensional space equipped with coordinates $x_1, ..., x_n$ is referred to as *the phase space* of the dynamical system (1.2), and points of this space are referred to as *phase points.* Thus, the phase space with coordinates $x_1, ..., x_n$ is the n-dimensional real Euclidean space $\mathbb{R}^n$. Now we are in the position to give a more precise definition of an invariant set of the dynamical system.

The set $M \subset \mathbb{R}^n$ is called an invariant set of system (1.2), if for every point $x^0 \in M$ we have $x(t) \in M$, $-\infty < t < +\infty$, where $x(t)$ is the (unique) solution of system (1.2) which satisfies the initial condition $x(0) = x^0$.

In the theory of differential equations thus defined invariant sets are often referred to as integral sets [81].

The solution $x(t, x^0, t_0)$ of system (1.2) which satisfies the initial condition $x(t_0) = x^0$ defined a *phase curve* in the phase space which passes through the point x^0.

A phase curve is, by definition, an invariant curve. The totality of phase curves, obtained by varying x^0 in the domain of permissible values in $\mathbb{R}^n$ is referred to as *phase portrait* of system (1.2).

Another corner-stony in the modern theory of dynamical systems is the notion of an attractor.

1.2.1. Attractors

A phase curve of system (1.2) may consist of a single point. Such a phase curve is called an *equilibrium state*. An equilibrium state must satisfy $dx/dt = 0$. Hence, the coordinates of the equilibrium states must be the roots of the system

$$F(x) = 0. \tag{1.4}$$

If $x = x^0$ is a single root of (1.4) (i.e. the Jacobian $(DF/Dx)_{x=x^0} \neq 0$), then the equilibrium state x^0 is said to be *simple*.

The equilibrium is stable, if $Re\lambda_k < 0$ for any root λ_k of the *characteristic equation*

$$\det(A - \lambda E) = 0, \tag{1.5}$$

where $A = (DF/Dx)_{x=x^0}$, and E is the unit matrix. A stable equilibrium is the simplest example of an attractor.

Along with equilibrium states system (1.2) can have isolated closed phase curves called *limit cycles*. So, a stable limit cycle can also serve as an attractor.

The structure of the phase space is well understood for two-dimensional systems ($n = 2$). They can have only two types of attractors: equilibrium states and limit cycles. For multidimensional ($n > 2$) systems there could exist other types of attractors, such as two-dimensional tori.

The attractors we have just mentioned are (geometrically) regular. It turns out, however, that a dynamical system can also have irregular ones usually referred to as *strange attractors*.

The term "strange attractor" was first introduced by Ruelle & Takens [88] who tried to describe the onset of turbulence, although the initial inspiration had come from the famous Lorenz's paper on thermal convection [45]. Lorenz was first to discover the existence of a stable deterministic non-periodic solution — strange attractor — in a concrete three-dimensional dynamical system with a quadratic nonlinearity (for details see Section 6.5).

By now there are known several "scenarios" of transition from a regular attractor to an irregular one. Those are the scenarios of Landau, Ruelle-Takens, Feigenbaum (period-doubling), Pomeau-Manneville (intermittency), etc. By a "scenario" we mean (after Eckman [22]) "one of the most likely possibilities" (see Chapter 6).

So far we have not introduced parameters into the systems (1.1)–(1.3), although, in reality they are always present. Therefore, let us now replace system (1.2) by the system

$$\frac{dx}{dt} = F(x,\gamma), \quad x = (x_1, ..., x_n); \quad F = (f_1, ..., f_n), \tag{1.6}$$

where γ is a vector of parameters.

"Transitions", we speak of in the scenarios, mean the passages of parameter γ through certain critical values called *bifurcational* values. The situation of a qualitative change in the behavior of system (1.6) that occurs for some critical parameter value $\gamma = \gamma_*$, is referred to as a *bifurcation* (see Chapter 6). Remark that Ruelle and Takens [88] did not give a precise definition of a strange attractor. Following Afraimovich [2] we say that

An attractor is strange if it is not a finite union of smooth manifolds..

Let us also give a general definition of an attractor.

For a set $V \subset \mathbb{R}^n$ and $t \in \mathbb{R}$ define the translation $V^t = x(t) : x(0) \in V$ of the set V by time t. We say that an (open) domain V is absorbing if $\bar{V}^t \subset V$ for all $t > 0$, where $\bar{V}$ is the closure of V.

Set $\mathbb{A}$ is called the maximal attractor in the absorbing domain V, if $\mathbb{A} = \cap_{t>0} V^t$.

it Set $\mathbb{A} \subset \mathbb{R}^n$ is an attractor if there exists an absorbing domain $V \supset \mathbb{A}$ in which $\mathbb{A}$ is the maximal attractor.

Along with attracting equilibrium states systems (1.6) may have other types of equilibrium states (for the classification of equilibrium states see [74], [5],[6], [9], [3]). Equilibrium states of the so-called *saddle type* are of special interest. Those are the equilibrium states of (1.2) for which there is an integer $0 < m < n$ such that m roots of the characteristic equation (1.5) have negative real parts, and the other $(n - m)$ roots [4] have positive real parts. Such an equilibrium has corresponding stable and unstable invariant manifolds of dimension m and $(n - m)$, respectively. This manifolds are called separatrices, since they can separate the phase space into domains with different behavior of phase curves. When $n = 2$ (thus, $m = 1$) a saddle has two (stable) separatrices asymptotically approaching the equilibrium as $t \to +\infty$, and two (unstable) separatrices which approach the equilibrium as $t \to -\infty$. The rest of the phase curves pass the saddle by. The separatrices separate the phase plane into sub-domains. When $n = 3$ a saddle has two one-dimensional separatrices and one two-dimensional. Clearly, the two-dimensional separatrix separates the phase space, while the one-dimensional do not.

The stable and unstable manifolds of a saddle may intersect each other or even coincide. If this is the case, then there exist double-asymptotic solutions, i.e.

[4] Roots of multiplicity r should here be counted r times, so that the total number of roots always equals the degree n of the equation (translator's note).

those asymptotically approaching the saddle equilibrium both as $t \to +\infty$ and as $t \to -\infty$. Poincaré called such solutions homoclinic, while those solutions that approach different equilibrium states (saddles) as $t \to +\infty$ and as $t \to -\infty$ he called heteroclinic.

A very interesting situation occurs when $n > 2$ and the stable and unstable separatrices intersect each other transversally (i.e. at an angle, not tangentially). *It has been discovered (see, e.g.* [96]), *that if this is the case, then in any* neighborhood of the double asymptotic phase curve there is an infinite (countable) set of saddle periodic solutions (limit cycles) and a very complicated structure. By "very complicated structure" we mean the presence of a, so-called, non-trivial hyperbolic set, the structure which Shil'nikov [96] called the *Poincaré homoclinic structure.* He showed [97] that if system (1.6) with $n > 2$ has a separatrix loop of a saddle-focus [5], then in a neighborhood of the loop there may exist the homoclinic structure. To ensure its existence certain conditions have to be verified.

In recent years people tend to use the words "chaos" or "deterministic chaos" instead of "complicated structure". So, in other words, one might say that the Shil'nikov's result is just another scenario of a transition to chaos: to determine the parameter values for which in system (1.6) there is a chaos, you have to look for the separatrix loop of a saddle-focus.

1.2.2. Invariant Tori

Attractors only exist in non-conservative systems. If we consider a conservative one, e.g. a Hamiltonian system

$$\begin{aligned} \frac{dq}{dt} &= \partial H(p,q)\partial p \quad q = (q_1, ..., q_m),\ p = (p_1, ..., p_m) \\ \frac{dp}{dt} &= -\frac{\partial H(p,q)}{\partial q}, \end{aligned} \tag{1.7}$$

we shall see that they possess equilibrium states of neutral type. For instance, for $m = 1$ those are the equilibrium states for which the roots of characteristic equation (1.5) are purely imaginary (Re $\lambda = 0$). An important characteristic of system (1.7) is the existence of the first integral ("energy" integral)

$$H(p,q) = h = \text{const.} \tag{1.8}$$

The function $H(p,q)$ is called the Hamiltonian function of the system (1.7) and q и p are called the generalized coordinates and impulses, respectively.

[5]The equilibrium is a saddle-focus (assuming $n = 3$) if (1.5) has two complex conjugate roots $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, with $b \neq 0$ and a real root λ_3 such that $a\lambda_3 < 0$.

As we mentioned earlier, the Hamiltonian systems originated from the celestial mechanics. The most significant results on their dynamics in post-Poincaré decades belong to Kolmogorov, Arnold and Mozer (the so-called KAM theory) [39],[7], [8], [69], [70] and basically state the following.

By introducing the new variables: $I = (I_1, ..., I_m)$ — action, and $\theta = (\theta_1, ..., \theta_m)$ — angle, one ca reduce the Hamiltonian function to the form $H = H_0(I)$. The Hamiltonian system is now immediately integrable. Its phase space (under very general conditions) splits into m-dimensional invariant tori $T^m = \{\theta_1, ..., \theta_m \bmod (2\pi)\}$ with quasi-periodic motions on them. The KAM theory states that in the perturbed system with Hamiltonian function $H(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta)$, where ε is a small parameter, a "majority" of invariant tori are preserved and only undergo a slight deformation.

KAM theory would have solved the problem of stability of the Solar system, if we only had $m = 2$. Indeed, in the case of two degrees of freedom ($m = 2$) every two-dimensional invariant torus T^2 separates the three-dimensional level set $H(I,\theta) = const$, and, thus a phase curve which is between two such tori at the moment $t = 0$ will forever stay "locked" between them (the action variables stay close to their initial values).

But since for the solar system $m > 2$, the situation is not that simple, and the KAM-theory does not solve the stability problem. Another subtle aspect of the problem has to do with the smallness of the parameter ε [29].

1.3. Discrete Dynamical Systems — Maps

If the independent variable (time) t varies in a discrete way, we come to a *discrete dynamical system*. Such systems are usually referred to as *maps* and, sometimes, *cascades*.

According to their origin the maps could be:

(a) obtained from an approximation of a differential equation by a difference equation;

(b) Poincaré maps of some cross-section (see Section 4.3.1), obtained by integrating some system of differential equations;

(c) defined by recurrent formulas.

Case (a) belongs to a special field of applied mathematics which studies computational methods. We shall be concerned most of all with the Poincaré maps and recurrent formulas. The discrete dynamical system defined by a map can be presented as follows:

$$X_{j+1} = f(X_j),\ j = 0, 1, 2, ...;\ X = (x_1, ..., x_n);\ f = (f_1, ..., f_n). \tag{1.9}$$

The map (1.9) will, for the sake of convenience, be denoted by T. Let us now assume that our map is one-to-one (homeomorphism, diffeomorphism). Such are the Poincaré maps generated by smooth systems of differential equations. Given the initial point X_0, we obtain the sequence

$$...X_{-n}, X_{-(n-1)}, ..., X_{-1}, X_0, X_1, ..., X_n, X_{n+1}, ...,$$

which is said to be a trajectory of map T. The trajectory can also be presented in the form

$$...T^{-n}X_0, T^{-(n-1)}X_0, ..., T^{-1}X_0, X_0, T^1X_0, ..., T^nX_0, ..., .$$

The map T may have fixed and periodic points.

Maps which do not have a well-defined inverse map (are not one-to-one) require a special treatment. Those are called irreversible maps or endomorphisms. Such maps (in two dimensions) are studied in the book of Mira, Gardini, Barugola and Cathala [52] with extensive use of the computer. In this book we shall discuss them in more detail in Chapter 7. Now we only mention that the endomorphisms normally give rise to fractals.

The point M is said to be periodic of period $p > 0$, if $T^pM = M$ and $T^nM \neq M$ for all $0 < n < p$. If $p = 1$, the point is said to be fixed. For $p > 1$ we have a cycle consisting of p periodic points.

In a manner similar to the analysis of equilibrium states one can classify fixed (periodic) points. In order to do so, one has to linearize the map T (T^p) at the fixed (periodic) point and find the eigenvalues (multiplicators) of the resulting Jacobi matrix. Let X_* be a fixed point of the map T, and let $A = (Df/DX)_{X=X_*}$ be the Jacobi matrix (the differential) of T at X_*. Then the *multiplicators* of T at X_* are the roots of the characteristic equation

$$\det(A - \mu E) = 0. \tag{1.10}$$

where E is the unit matrix of appropriate dimension. The fixed point X_* is stable, if $|\mu_j| < 1$, $j = 1, .., n$, and unstable otherwise. The saddle (hyperbolic) points, i.e. those for which some multiplicators have absolute value greater than 1 and the rest — smaller than 1, play an important role in the study of the dynamics of T. If X_* is a periodic point for T with the period p, then X_* is stable (unstable, saddle) for T, if and only if X_* is a stable (unstable, saddle) fixed point for T^p. Like the equilibrium states, the saddle fixed (periodic) points have their separatrices. Along with regular attractors, the maps may have strange attractors.

Maps (1.9) are widely studied mostly for $n = 1$ and $n = 2$. A detailed discussion of maps can be found in Chapters 4–7.