with the estimation of the number of equilibrium states and periodic orbits by means of the topological invariants of manifolds. Later, Palis and Smale [106, 102] proved the following theorem.

Theorem 7.10. (Palis and Smale) *Morse–Smale systems are structurally stable.*

This theorem was proven for systems whose phase space is a compact, smooth manifold. It holds for our case as well, if suppose that the boundary ∂G is a smooth $(n-1)$-sphere without contact, through which a trajectory goes inwards of G in the continuous case, or that $X(G) \subset G \backslash \partial G$ in the discrete case.

7.6. Some properties of Morse–Smale systems

Comparing the Andronov–Pontryagin theorem with the definition of Morse–Smale systems, one can see that the last ones are quite similar to rough systems on the plane and are, in essence, their high-dimensional generalization. Like the Andronov–Pontryagin theorem, the Palis–Smale Theorem 7.10 yields sufficient conditions for roughness. Axiom 2 in Definition 7.9 may be viewed as a natural necessary condition. In contrast, Axiom 1 has nothing to do with the problem of structural stability but it restricts rather severely the class of systems under consideration, and suppresses many hidden opportunities which saddle equilibria, and periodic orbits can exhibit in dimensions higher than two.

For example, the following theorem shows that a Morse–Smale system cannot have a homoclinic trajectory to a saddle periodic orbit.

Theorem 7.11. *Let L be a saddle periodic orbit, and let Γ be its homoclinic trajectory along which W^s_L and W^u_L intersect transversely. Then, any small neighborhood of $L \cup \Gamma$ contains infinitely many saddle periodic orbits.*

Proof. Take a small cross-section S to L and consider the local Poincaré map $T_0 : S \to S$. The point $O = S \cap L$ is the saddle fixed point of T_0. Let us introduce the coordinates (x, y) on S near O such that the local unstable manifold of O is $x = 0$ and the local stable manifold is $y = 0$ (thus, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ where $\dim W^s = m$, $\dim W^u = p$). Let $M^-(0, y^-) \in W^u_{\text{loc}}$ and $M^+(x^+, 0) \in W^s_{\text{loc}}$ be two points of intersection of the homoclinic orbit Γ with S. The flow near the piece of Γ between M^- and M^+ defines a map T_1 from a small neighborhood Π^- of M^- onto a small neighborhood Π^+ of M^+ on S. This map can be written in the form

$$\begin{aligned}\bar{x}^0 - x^+ &= ax^1 + b(y^1 - y^-) + \cdots\,, \\ \bar{y}^0 &= cx^1 + d(y^1 - y^-) + \cdots\,,\end{aligned} \tag{7.6.1}$$

where the ellipsis stand for nonlinear terms; (x^0, y^0) refer to a small neighborhood of M^+ and (x^1, y^1) refer to a small neighborhood of M^-. Observe that the image $T_1 W^u_{\rm loc}$ is tangent at M^+ to the p-dimensional plane defined by the parametric equation

$$x^0 - x^+ = bu\,, \quad y^0 = du\,,$$

where $u \in \mathbb{R}^p$. By assumption, this hyperplane must be transverse to $y = 0$ which means that

$$|d| \neq 0\,. \tag{7.6.2}$$

It was shown in Sec. 3.7 that for any sufficiently large k there are points in Π^+ whose kth iteration by the local map T_0 lies in Π^-. The set of such points is "a horizontal strip" σ^0_k. As $k \to \infty$, the horizontal strips accumulate at $W^s_{\rm loc} \cap \Pi^+$. The map T^k_0 contracts the strip in the x-direction and stretches it in the y-direction, so that the images $T^k_0 \sigma^0_k$ ("the vertical strips" σ^1_k) accumulate at $W^u_{\rm loc} \cap \Pi^-$. It is geometrically evident (see Fig. 7.5.3) that due to the transversality of $T_1 W^u_{\rm loc}$ to $W^s_{\rm loc}$, the image $T_1 T^k_0 \sigma^0_k$ intersects σ^0_k "properly" for any sufficiently large k, so that the map $T_1 T^k_0|_{\sigma^0_k}$ is *a saddle map* in the sense of Sec. 3.15. By Theorem 3.28, a saddle map has a saddle fixed point. Since both maps T_1 and T_0 are defined by the orbits of the flow, the fixed point of $T_1 T^k_0$ corresponds to a periodic orbit of the system (it intersects S exactly k times, the first time in Π^+ and the last time in Π^-). Taking different k we will obtain different periodic orbits. Thus, to prove the theorem we must confirm by computation that the maps $T_1 T^k_0$ are of the saddle type for all sufficiently large k.

By Lemmas 3.3 and 3.4, there exist functions ξ_k, η_k which uniformly tend to zero along with their derivatives as $k \to \infty$, such that a point $M^0(x^0, y^0)$ is mapped into a point $M^1(x^1, y^1)$ by T^k_0 if, and only if,

$$\begin{aligned}x^1 &= \xi_k(x^0, y^1)\,, \\ y^0 &= \eta_k(x^0, y^1)\,.\end{aligned} \tag{7.6.3}$$

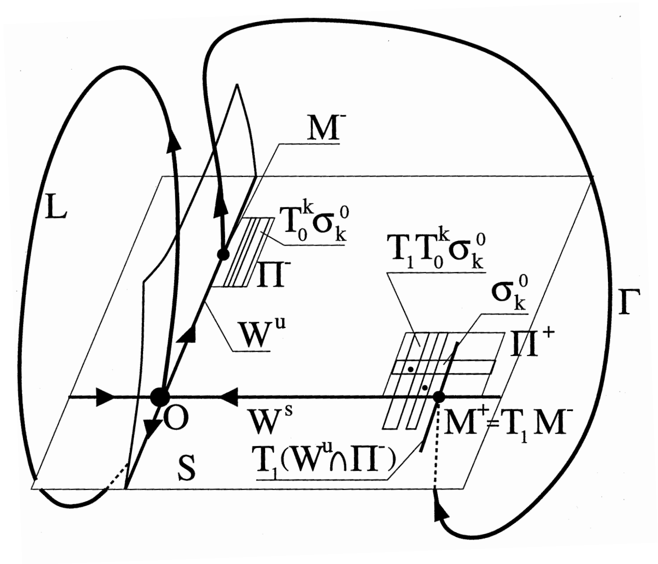

Fig. 7.5.3. The Poincaré map near a transverse homoclinic orbit.

Note that due to (7.6.2), the second equation in (7.6.1) can be solved for y^1, with a sufficiently small x^1 and $\bar{y}^0$:

$$y^1 - y^- = d^{-1}\bar{y}^0 - d^{-1}cx^1 + \cdots ,$$

where the dots stand for small nonlinear terms. Comparing this with the first equation of (7.6.3) we obtain, for sufficiently large k,

$$y^1 = y^- + F_k(\bar{y}^0, x^0) , \tag{7.6.4}$$

where F_k is a smooth function such that $F_k(0,0) = 0$, and the derivative of F_k with respect to x^0 tends uniformly to zero as $k \to \infty$ (we use that $\frac{\partial \xi_k}{\partial(x,y)} \to 0$ as $k \to \infty$).

Now, for all sufficiently large k the map $T_1 T_0^k$ can be written as

$$\begin{aligned} \bar{x} &= x^+ + G_k(\bar{y}, x) , \\ y &= \eta_k(x, y^- + F_k(\bar{y}, x)) , \end{aligned} \tag{7.6.5}$$

where we suppress the upper index 0; here G denotes a smooth function such that $G_k(0,0) = 0$, and the derivative of G_k with respect to x tends uniformly to zero as $k \to \infty$. This is a *cross-form* of the map $T_1 T_0^k$ in the sense of Sec. 3.15 (the spaces D_1 and D_2 in Definition 3.6 are small convex neighborhoods of x^+ in the x-space and zero in the y-space, respectively). Since the derivatives of η_k with respect to all variables, and the derivative of G_k with respect to x tend uniformly to zero as $k \to \infty$, it is easy to see that the map $T_1 T_0^k$ fits Definition 3.7 of the saddle map for all sufficiently large k,[6] so Theorem 3.28 on the fixed point is applicable here. This completes the proof.

The above proof can be easily translated into the language of diffeomorphisms with a fixed point having a transverse homoclinic trajectory. It also covers the case of a periodic point with a homoclinic trajectory. In the last case one should consider the qth iteration of the original diffeomorphism, where q is the period.

In essence, the above proof is a close repetition of that suggested by L. Shilnikov [131]. It allows one to liberate from the axiom stipulating the absence of homoclinic trajectories in Morse–Smale systems originally postulated by Smale.

Note that a transverse homoclinic orbit is, obviously, preserved under small smooth perturbations of the system. Therefore, Theorem 7.11 implies that when a transverse homoclinic exists, any close system is away from the Morse–Smale class. This gives us a robust and simple indicator for detecting the complex dynamics. By now, the presence of transverse homoclinics is regarded as a universal criterion of chaos.

As before, we will regard equilibrium states and periodic orbits as equal objects and denote them by Q for uniformity.

[6]The condition to check is

$$\|P'_x\|_\circ < 1\,, \quad \|Q'_y\|_\circ < 1\,,$$
$$\|P'_y\|_\circ \|Q'_x\|_\circ < (1 - \|P'_x\|_\circ)(1 - \|Q'_y\|_\circ)\,,$$

where (P, Q) are the right-hand sides of the cross-map:

$$\bar{x} = P(x, \bar{y})\,, \quad y = Q(x, \bar{y})\,.$$

Essentially, this means that the cross-map is contracting in a suitable norm, so the map itself is strongly expanding in the y-direction and strongly contracting in the x-direction.

Let us introduce the following notation: we write $Q_i \leq Q_j$ if $W^s_{Q_i} \cap W^u_{Q_j} \neq \emptyset$, in particular $Q_i \leq Q_i$. If $(W^s_{Q_i} \setminus Q_i) \cap (W^u_{Q_j} \setminus Q_j) \neq \emptyset$ then we will write $Q_i < Q_j$. We will say that $Q_{k_1}, \ldots, Q_{k_l}$ form *a chain* if

$$Q_{k_1} < \cdots < Q_{k_l} \,. \tag{7.6.6}$$

If the first and the last members of the chain are equal ($Q_{k_1} = Q_{k_l}$), then the chain (7.6.6) is called *a cycle*.

It may be proved that the relation "$\leq$" defines a partial order on the set of non-wandering orbits of a Morse–Smale system. An important result is:

Theorem 7.12. *There are no cycles in Morse–Smale systems.*

First of all, observe that there cannot exist cycles like $Q_0 < Q_0$ because homoclinic trajectories are not admissible in Morse–Smale systems. Also, it follows from the transversality condition [see (7.5.4)] that a cycle cannot contain equilibrium states; neither can it include periodic orbits of different topological types.

Thus, only one hypothesis remains; namely when the cycle

$$L_0 < L_1 < \cdots < L_k < L_0$$

is composed of periodic orbits of the same topological type. Consider the chain

$$L_{k-1} < L_k < L_0$$

and let $W^s_{L_{k-1}}$ intersect $W^u_{L_k}$ at a point x_0. By virtue of the λ-lemma (see Sec. 3.7), we may claim that since $W^s_{L_k}$ intersects $W^u_{L_0}$ transversely, in any small neighborhood U of the point x_0, there is a countable set of smooth pieces of $W^u_{L_0}$ converging to $W^u_{L_k} \cap U$. Since $W^s_{L_{k-1}}$ intersects $W^u_{L_k}$ transversely, it follows that $W^s_{L_{k-1}}$ intersects these pieces of $W^u_{L_0}$. Therefore

$$L_{k-1} < L_0 \,.$$

Continuing inductively, we obtain

$$L_0 < L_1 < L_0 \,,$$

and hence

$$L_0 < L_0\,,$$

i.e. L_0 has a homoclinic trajectory. This contradicts Theorem 7.11.

The fact that there are a finite number of non-wandering trajectories in Morse–Smale systems implies that any chain has a finite length which does not exceed the total number of non-wandering trajectories. Moreover, a maximal chain can end *only* at a stable equilibrium state or a periodic orbit.

It follows from the above arguments that one may introduce an oriented graph for each Morse–Smale system. Its vertices are equilibrium states and periodic orbits, with the topological type assigned. The edges of the graph are oriented in a decreasing way in accordance with the order $<$. Namely, the vertex Q_i is connected with the vertex Q_j by an edge if and only if $Q_i > Q_j$ and there is no Q_k such that $Q_i > Q_k > Q_j$. Such a graded graph is called *a phase diagram*. The phase diagram for a Morse–Smale diffeomorphism can be introduced in an analogous way. The vertices are fixed points and cycles, with their local characteristics specified.

It is clear that the phase diagram is an invariant of topological equivalence of Morse–Smale systems.

However, it is not a complete invariant, generally speaking. For example, it contains no information on the number of orbits of intersection of stable and unstable manifold of saddle trajectories.

Among all heteroclinic trajectories one may select some *special* ones which play a central role.

Definition 7.12. *A heteroclinic trajectory* Γ *is said to be special if there exists a neighborhood* U *of its closure* $\bar{\Gamma}$ *which contains no other heteroclinic trajectories but* Γ.

It is obvious that all heteroclinic trajectories of three-dimensional Morse–Smale flows are that special. This is also true for two-dimensional diffeomorphisms.

The principal feature of Morse–Smale systems which distinguishes them from Andronov–Pontryagin systems is that the former may have infinitely many special heteroclinic trajectories. As an example, let us consider a two-dimensional diffeomorphism with three fixed points of the saddle type denoted by O_1, O and O_2. Suppose that $W^s_{O_1} \cap W^u_O \neq \emptyset$ and $W^s_O \cap W^u_{O_2} \neq \emptyset$, the

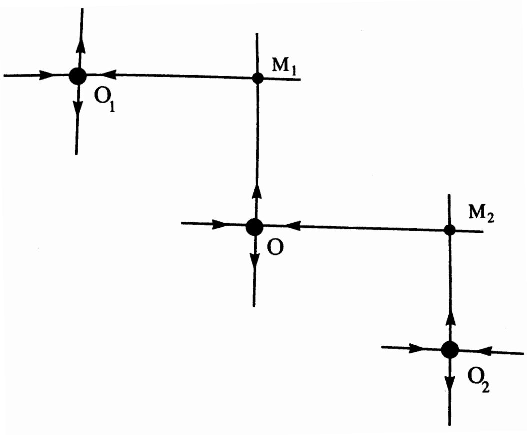

Fig. 7.6.1. Hierarchic intersections of the manifolds of fixed points at the heteroclinic points M_1 and M_2.

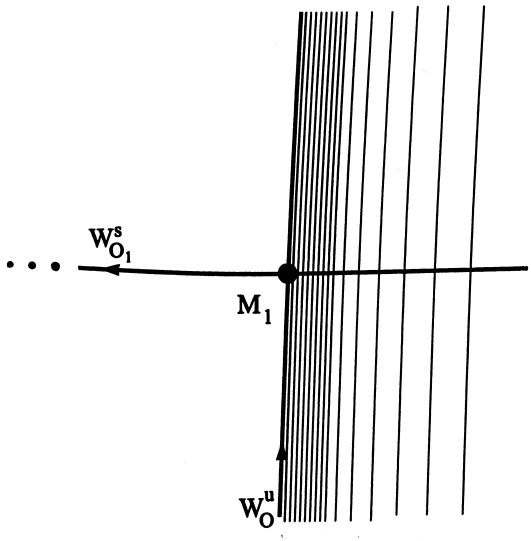

Fig. 7.6.2. A neighborhood of the point M_1: the successive images of $W^u_{0_2}$ accumulate to W^u_0 Fig. 7.6.1.

corresponding scheme is depicted in Fig. 7.6.1. Here, $M_1 \in W^s_{O_1} \cap W^u_O$ and $M_2 \in W^s_O \cap W^u_{O_2}$, i.e. they are heteroclinic points.

Let us now apply the λ-lemma (see Sec. 3.7). Choose a small neighborhood U of the point M_1. It follows that the intersection $U \cap W^u_{O_2}$ consists of a countable set of curves l_k $(k = 1, \dots, \infty)$ accumulating smoothly to W^u_O, as shown in Fig. 7.6.2. As $W^s_{O_1}$ and W^u_O intersect each other transversely, then $W^s_{O_1}$ intersects each l_k at the points M_k starting from some number k_0. The points M_k are heteroclinic too and correspond to different heteroclinic trajectories which have O_1 and O_2 as an α-limit and an ω-limit points, respectively.

An analogous picture takes place in the case of three-dimensional flows possessing the chain $Q_1 < Q < Q_2$ where Q denotes a saddle periodic orbit, and Q_1 and Q_2 stand for either saddle equilibrium states or periodic orbits.

Heteroclinic trajectories are no longer all special in higher-dimensional case. A heteroclinic trajectory $\Gamma \subset W^s_{Q_1} \cap W^u_{Q_2}$ is special only if the dimension of the forming intersection is equal to 1. We will assign to Γ a type $(\dim W^s_{Q_1}, \dim W^u_{Q_2})$. It is clear that if $\dim W^s_{Q_1} = m+1$, then the type of the special trajectory Γ is $(m+1, n-m)$, where n is the dimension of the phase space.

Let us consider a Morse–Smale flow having two non-wandering motions Q_1 and Q_2. Let dim $W^s_{Q_1} = m+1$ and dim $W^u_{Q_2} = n-m$. Suppose next that $W^s_{Q_1} \cap W^u_{Q_2} \neq \emptyset$, i.e. $Q_1 \leq Q_2$.

Theorem 7.13. (Afraimovich and Shilnikov [2]) *The intersection* $W^s_{Q_1} \cap W^u_{Q_2}$ *possesses infinitely many heteroclinic trajectories if, and only if the closure* $\overline{W^s_{Q_1} \cap W^u_{Q_2}}$ *contains a periodic orbit* L *of type* $(m+1, n-m)$, *other than* Q_1 *and* Q_2.

It follows from the proof of this theorem that W^u_L must intersect $W^s_{Q_1}$ transversely and W^s_L must intersect $W^u_{Q_2}$ transversely. Observe also that all trajectories in $W^s_{Q_1} \cap W^u_{Q_2}$ are special.

An analogous statement holds for Morse–Smale diffeomorphisms.

Let us denote by N_{m+1} the set of all special trajectories of type $(m+1, n-m)$ and their limiting non-wandering motions. In the general case, N_{m+1} consists of a finite number of connected components $N^{(1)}_{m+1}, \dots, N^{(k)}_{m+1}$. It was also shown in [2] that the set of all trajectories $N^{(l)}_{m+1}$, where $1 \leq l \leq k$, is in a one-to-one correspondence with the set of all trajectories of some symbolic system with a finite number of states. Generally speaking, symbolic dynamics was historically created in connection with the description of systems with complex dynamics. Nevertheless, it has turned out that it can be effectively applied to Morse–Smale systems possessing a countable number of special heteroclinic trajectories as well.