the attempts of analysts. Computers made progress possible again. Recent numerical work has shown that even *few*-body systems show irreversible behavior, on the average, even with rigorously time-reversible motion equations. The irreversibility emerges with great clarity and precision when the small-system results are time averaged.

Computers made it possible to simulate both reversible mechanics and irreversible flows. In the latter case it was necessary to impose boundary conditions or constraints, driving the system from equilibrium. Heat and work had to be incorporated explicitly into the programming. Handily, all this could be done without sacrificing the time reversibility of the underlying equations!

1.8 Example Problems

To illustrate the concepts of time reversibility and chaos we consider here three examples. The first is a two-dimensional area-preserving map. It is a caricature of equilibrium flows obeying Liouville's incompressible theorem. The second is a three-dimensional continuous flow, but with *discontinuous* forces and a simple phase-space structure. The last is a three-dimensional flow with continuous forces and a *complicated* phase-space structure. All three of these problems can exhibit chaotic behavior, with small changes in initial conditions growing exponentially with time. All three are relatively easy to simulate and to visualize. Each is a building block in creating an understanding of the nonequilibrium systems which are emphasized in the following Chapters of the book. The underlying background in mechanics and numerical algorithms, required to generate numerical solutions for the last two problems, is given in Chapter 2.

1.8.1 *Equilibrium Baker Map*

If N coordinates suffice to represent a system's configuration then the configuration at any fixed time is represented by a single point in the corresponding N-dimensional space. The time development of that point defines a one-dimensional "trajectory"—a line—in that same space. Imagine the repeated intersections of such a trajectory with some fixed $(N-1)$-dimensional surface embedded in the N-dimensional configuration space. The successive intersections with such a surface provide a "Poincaré Section" defining a "mapping" on the surface coordinates, which links each intersection to the next. Such "maps" can be viewed as simplified caricatures of flow problems. Their advantage is a reduction (by one) in the number of coordinates which has to be considered.

Because a *one*-dimensional reversible map can only alternate between two values of the dependent variable, the simplest interesting *reversible* map is necessarily *two*-dimensional. Such a map is analogous to an ordered series of snapshots, each related to the previous by an integration of the motion equations over the time interval Δt by which the snapshots are separated. More complicated maps, with irregular time intervals, correspond to the crossings of particular surfaces in the phase space.

If the underlying motion equations are reversible, the map must likewise be reversible. The two-dimensional Baker Map which we consider here is often used to exhibit chaos, "sensitive dependence on initial conditions", because the separation of two nearby points increases exponentially, in a particular direction, the "unstable" direction. Provided that the initial point is chosen sufficiently randomly, with irrational coordinates for instance, the time-reversible Baker Map shown in Figure 1.1 eventually provides an "ergodic" coverage (coming arbitrarily close to all the points) of the 2×2 square, whether the map goes forward or backward in time. In both directions the map has the same form, but with x and y permuted in the backward map:

$$[-1 < x < +1 \; ; \; 0 < y < +1] \longrightarrow \{x, y\}' = \{(x+1)/2, (2y-1)\};$$

$$[-1 < x < +1 \; ; \; -1 < y < 0] \longrightarrow \{x, y\}' = \{(x-1)/2, (2y+1)\}.$$

This map *reduces* point-to-point separations in the x direction and increases them in the y direction, in both cases by a factor 2. The map is said to be "Lyapunov unstable" because small separation differences in the y direction double at each iteration, corresponding to a maximum Lyapunov exponent of $\ln 2$:

$$\delta y_{n+1} = 2\delta y_n \longrightarrow \lambda_1 \equiv \ln \langle (\delta y_{n+1}/\delta y_n) \rangle = \ln 2.$$

To attain a fixed level of precision in specifying the future iterates of an initial point, additional information would have to be added, in the y direction, and could be discarded, in the x direction, one binary bit per iteration.

Overall, the "equilibrium" Baker Map conserves area, with the stretching (in the y direction) exactly compensating for the shrinking (in the x direction). Evidently the action of the map is "sensitive" to small differences in the stretching direction, so that ***not even in principle*** could the "true" motion be followed for long. This Baker Map is a rough caricature of motions governed by Lyapunov-unstable Hamiltonian mechanics.

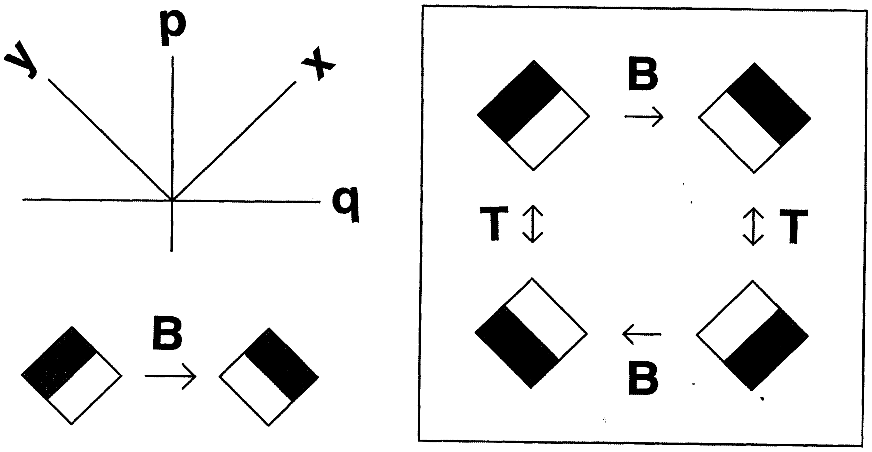

Figure 1.1: The Equilibrium Baker Map, B, in the usual (x, y) coordinates, is shown at the lower left. The mapping B relates the "new" coordinates to the old ones, $B_{xy}(x, y) = (x', y')$. The *rotated* map, $B_{qp}(q, p) = (q', p')$, in $(q, p) = \sqrt{\frac{1}{2}}(x - y, x + y)$ coordinates, is shown at the right. The time-reversal operation T_{qp} shown in the Figure corresponds to changing the sign of the "momentum" p.

When the exponential instability associated with stretching is represented by a limited-precision computer, it can give rise to a fixed y coordinate, at $y = \pm 1$, and a fixed x coordinate, $x = \pm 1$, so that the resulting "fixed point" repeats forever. Likewise, there are particular initial conditions which give rise to (unstable) periodic cycles of two, three, four, ... iterations. For example, the twice-iterated Baker Map has a cycle connecting the two (x, y) points $(-1/3, +1/3)$ and $(+1/3, -1/3)$. On the other hand, adding a small amount of random noise, in the eighth, tenth, or twelfth significant figure, is enough to destroy these artificial cycles.

This equilibrium Baker Map, though simple, already illustrates a disquieting feature, due to its singular nature along the line $y = 0$: any *rational* value of y, of the form $\pm n/2^m$, is eventually mapped to the singular line. For any fixed m these solutions, like those of the periodic cycles, have measure zero in the two-dimensional space. General *irrational* points are instead mapped in an "ergodic" manner, eventually coming arbitrarily close to any (x, y) point in the square. Despite this uniform coverage of the square, the Baker Map is a very poor random number generator. In both the (x, y) and the rotated (q, p) representations it is easy to show that the products of successive iterates are strongly correlated:

$$\langle xx' \rangle = 1/6 \ ; \ \langle yy' \rangle = 2/3 \ ; \ \langle qq' \rangle = \langle pp' \rangle = 5/12.$$

All these correlations would vanish if the (x, y) points or the rotated (q, p) points could truly be chosen *randomly.*

The odd sensitivity to infinitesimal insignificant "information" which has no significance, being infinitesimal in scale, can plainly have nothing to do with physics. The sensitivity is a consequence of the vagaries of the real number system, in which any two rational numbers, no matter how close together, are separated by an infinite continuum of irrationals as well as a set with the lesser cardinality, $\aleph_0$ in Cantor's theory of infinite sets, of rational numbers. Because the Baker Map is *so* simple some computer programs, on some computers, will not generate sufficient internal noise to avoid settling onto one of the Map's fixed points at $(x, y) = \pm(1, 1)$. In an accurate calculation this could not happen because these fixed points are unstable, with displacements in the y direction doubling at each iteration.

The disparity in evolution, between the rational periodic orbits and general irrational initial points, seems less serious if we consider exactly the same map, expressed now in terms of the horizontal and vertical coordinates, $\{q, p\}$. The motivation for this rotation is the physicist's notion of time reversibility: coordinate values $\{q\}$ should be traced out backward in time simply by (i) reversing the signs of the corresponding momenta $\{p\}$ and (ii) choosing the right initial conditions. The rotated (q, p) coordinates are related to the original (x, y) coordinates by a 45^o clockwise rotation:

$$q \equiv (x - y)/\sqrt{2} \ ; \ p \equiv (x + y)/\sqrt{2}.$$

Thus the periodic cycle of the twice-iterated map links the two (q,p) points $(\pm\sqrt{2}/3,0)$. In the new (q,p) coordinate system the map satisfies the usual physicists' expectation for a time-reversible map:

$$B_{qp}(q,p) = (q',p') \longrightarrow B_{qp}(q',-p') = (q,-p).$$

Explicitly, this rotated Baker's Map has the piecewise-linear, but *irrational* form:

$$q < p \longrightarrow \{q,p\}' = \{+(5q/4) - (3p/4) + \sqrt{9/8}, -(3q/4) + (5p/4) - \sqrt{1/8}\};$$

$$p < q \longrightarrow \{q,p\}' = \{+(5q/4) - (3p/4) - \sqrt{9/8}, -(3q/4) + (5p/4) + \sqrt{1/8}\}.$$

See Figure 1.2 for a sequence of points generated using this map. We will come back to simple mappings again, in discussing dissipative analogs of the present area-preserving conservative Baker Map. Though simple, these maps have considerable pedagogical value.

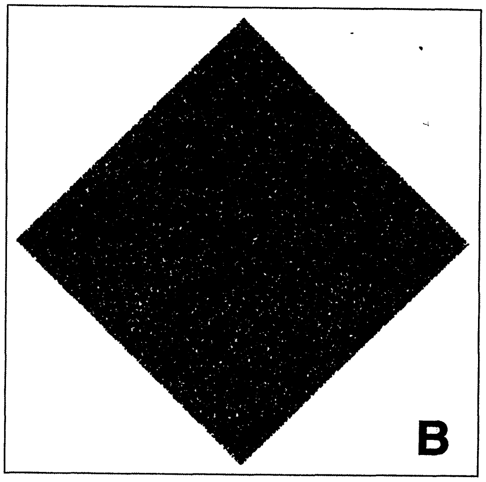

Figure 1.2: 100,000 points generated by the (q,p) Baker Map beginning with initial point at (0.50,0.00).

Evidently a probability density $f(x,y)$ which is constant is preserved exactly by the Baker Map. Each area element $dxdy$ is mapped to a new element with exactly the same area, $dx'dy' = (dx/2)2dy$. Thus the constant-density solution $f(-1 < x < +1, -1 < y < +1) \equiv 1/4$ is *stationary.* Apart from the lines with special rational y coordinates, $y = n/2^m$, along which