2. Model: The Creutz Cellular Automaton

In general, n binary bits are associated with each site of the lattice. The value for each site is determined from its value and those of its nearest neighbors at the previous time step. The updating rule, which defines a deterministic cellular automaton, is as follows: of the n binary bits on each site, the first one is the Ising spin B_i. Its value may be "0" or "1." The Ising spin energy (internal energy) of the lattice H_I, is given (in units of the nearest neighbor coupling constant J) by

$$H_I = -\sum_{\langle ij \rangle} S_i S_j \,, \tag{1}$$

where $S_i = 2B_i - 1$, and $\langle ij \rangle$ denotes the sum over all nearest neighbor pairs of sites. The next $n-2$ bits are for the momentum variable conjugate to the spin (the demon). These $n-2$ bits form an integer which can take on the values within the interval $(0,\ N = \sum_{i=1}^{n-2} 2^{i-1})$. The kinetic energy (in units of J) associated with the demon can take on four times these integer values. The total energy

$$H = H_I + H_K \tag{2}$$

is conserved; here H_K is the kinetic energy of the lattice. For a given total energy, the system temperature T (in units of J/k_B where k_B is the Boltzmann constant) is obtained from the average value of the kinetic energy of a demon,

$$\langle E_D \rangle = \frac{\sum_{m=0}^{N} (4m) e^{(-4m)/T}}{\sum_{m=0}^{N} e^{(-4m)/T}} \,. \tag{3}$$

The nth bit provides a checkerboard style updating, and so it allows the simulation of the Ising model on a cellular automaton. The black sites of the checkerboard are updated, and then their colour is changed into white; the white sites are changed into black without being updated.

The updating rules for the spin and the momentum variables are as follows: for a site to be updated, its spin is flipped and the change in the Ising energy (internal energy) H_I, is calculated. If this energy change is transferable to or from the momentum variable associated with this site, such that the total energy H, is conserved, then this change is done and the momentum is appropriately changed. Otherwise the spin and the momentum are not changed.

As the initial configuration all the spins are taken ordered (up or down). The initial kinetic energy is given to the lattice via the appropriate bits of the momentum variables in the white sites, randomly.

Simulations are carried out on simple hypercubic lattices L^d of dimensionality d and linear dimension L with periodic boundary conditions.

Some features of the model are as follows:

(1) It is reversible[19] like Q2R cellular automaton[8] for which this was shown experimentally.[18]

(2) It is not ergodic[19] like Q2R cellular automaton[8,12]; but at a given total energy, the effect of the possible spin configurations that cannot be reached, on the results for various quantities can be understood only by computer experiments.

Some other features of the model, arising mainly in the computer experiments, are as follows:

(3) Because of the checkerboard-style updating, only the lattices with the even values of the linear dimension $L = 2, 4, 6, 8, \ldots$ can be simulated.

(4) The number of energy levels of the demon determines if the simulation for a given lattice L^d is possible, and if so, the maximum value of the total energy after which the simulation is not possible. For example, the simulation of the four-dimensional Ising model with the two-bit demons (four energy levels) is not possible, because the maximum kinetic energy of a two-bit demon is 12 (in units of J), however at least 16 is needed to flip a spin at its initial configuration (all the spins down). Thus at least three-bit demons are necessary for simulating the four-dimensional Ising model. The simulation of the seven-dimensional Ising model with the three-bit demons (eight energy levels) is possible up to a total energy of ($H = 6.851318$) ($T = 12.557$) for $L = 4$ (Fig. 1).[22]

(5) The way the initial kinetic energy is given to the lattice, also determines the maximum total energy after which the simulation is not possible. For example, the simulation of the four-dimensional Ising model with the three-bit demons is possible up to a total energy of ($H = 3.625$) ($T = 6.633$) for $L = 4$, if the initial kinetic energy is given to the lattice via the third bits of the momentum variables (Fig. 2).[24]

(6) There is a lowest total energy (temperature) below which the Creutz cellular automaton does not simulate the Ising model. This temperature is, in terms of the reduced temperature, $\epsilon(L) \simeq 0.09$ for all $2 \leq d \leq 8$ and for all $L \geq 4$ when $T < T_c^{\chi}(L)$. This temperature reveals itself by the start of distortions in the curve(s), e.g., of the specific heat, the magnetization, and the magnetic susceptibility [Fig. 3(d)].[22]

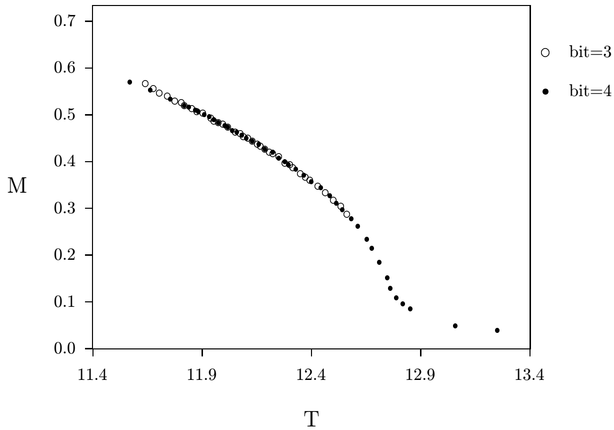


Fig. 1. The temperature dependence of the order parameter (M) of the seven-dimensional Ising model computed with the three- and four-bit demons for the lattice with $L = 4$. The simulation lasts 6×10^4 sweeps.

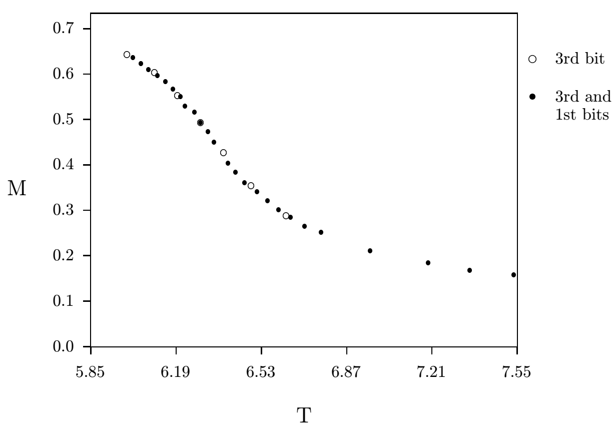
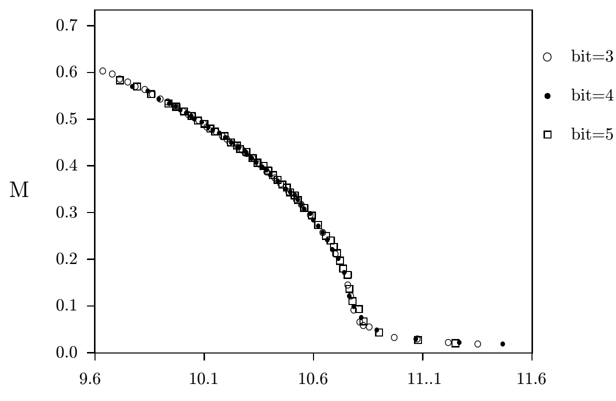


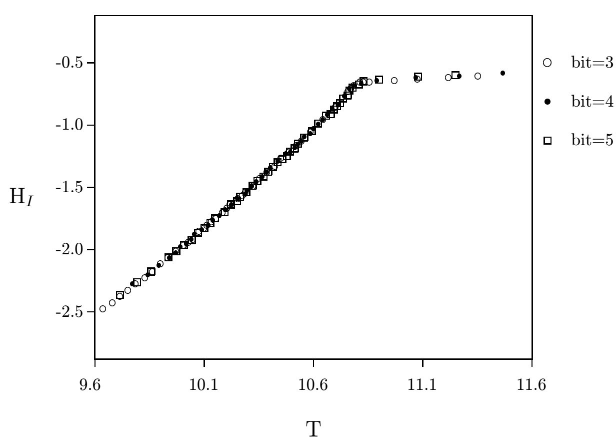
Fig. 2. The temperature dependence of the order parameter (M) of the four-dimensional Ising model computed with the three-bit demons for the lattice with $L = 4$, by giving the initial kinetic energy to the lattice via the third bit of the demon and via the third and first bits. The simulation lasts 9.6×10^5 sweeps.

(7) The number of energy levels of a demon almost do not affect the values of the magnetization and the internal energy [Figs. 3(a) and 3(b)].[22] However, the magnetic susceptibility (the fluctuations of the magnetization) and the specific heat (the fluctuations of the internal energy) are affected [Figs. 3(c)

and 3(d)],[22] and they converge to their limiting values with increasing number of energy levels. For the curves of these quantities, two temperature regions can be distinguished: for the magnetic susceptibility, one of these regions is the neighborhood of the critical temperature $T_c^{\chi}(L)$ where the

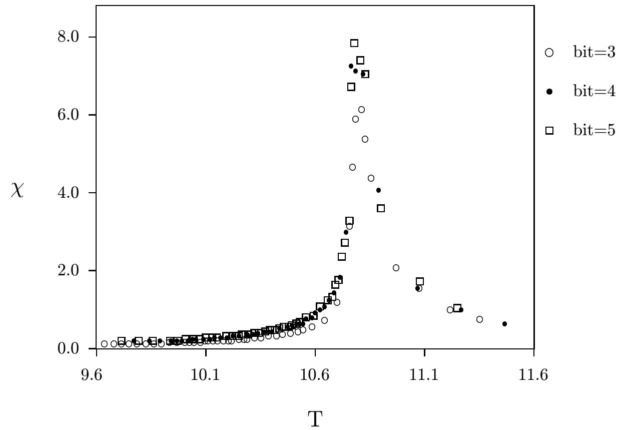


(a)

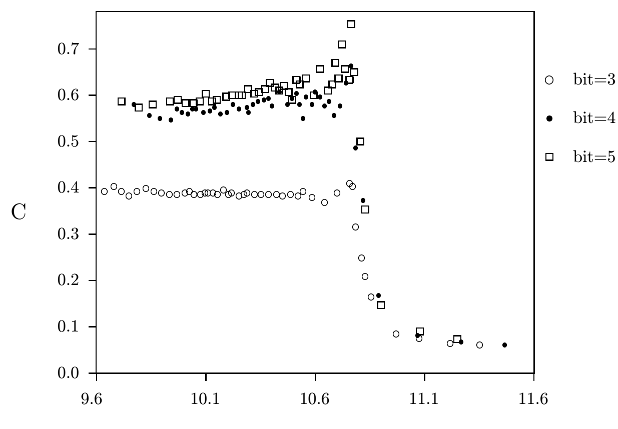


(b)

Fig. 3. The temperature dependence of (a) the order parameter (M), (b) the internal energy (H_I), (c) the magnetic susceptibility (χ), and (d) the specific heat (C, in units of k_B) of the six-dimensional Ising model computed with the three- to five-bit demons for the lattice with $L = 6$. The simulation lasts 6×10^4 sweeps.



(c)



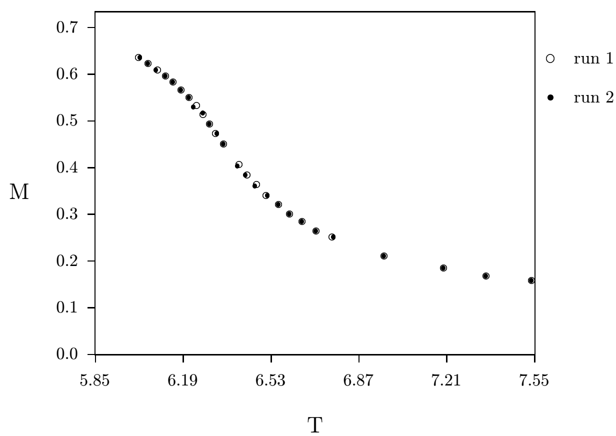
(d)

Fig. 3. (*Continued*).

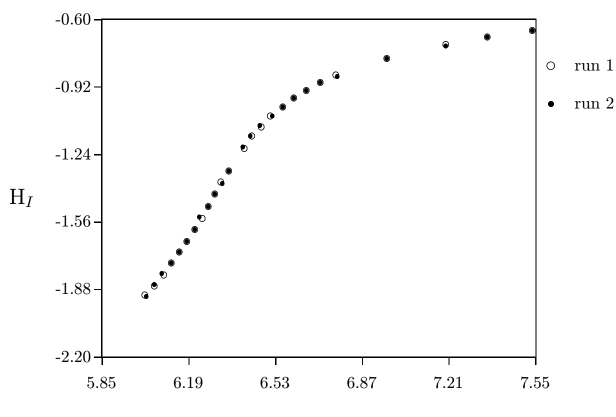
effect is more pronounced, and the other one is away from it; for the specific heat, one of these regions is $T \leq T_c^C(L)$ where the effect is more pronounced, and the other one is $T > T_c^C(L)$.

(8) Irrespective of the number of bits used in the simulations of a finite-size lattice L^d, the maxima for the magnetic susceptibility and the specific heat occur at their respective temperatures within the error limits, ± 0.02. This

conclusion is supported by the invariance, with the number of demon bits, of the order parameter and the internal energy curves, and hence, their inflection points which correspond to their critical temperatures (Fig. 3).[22]

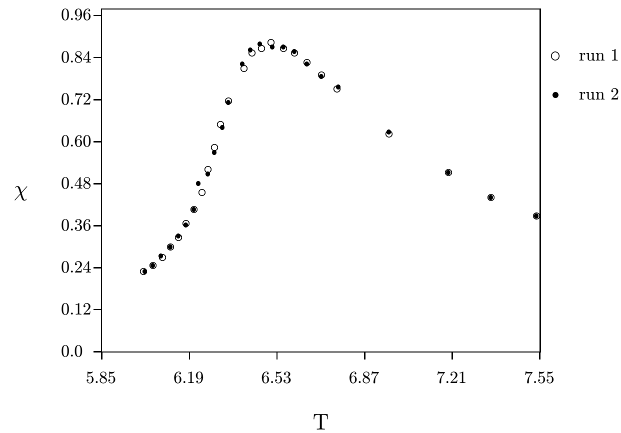


(a)

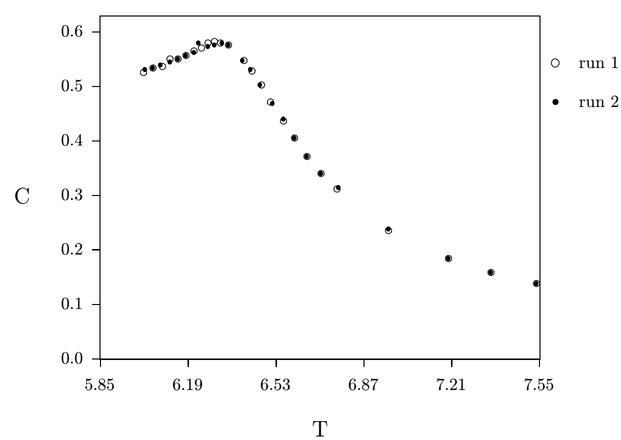


(b)

Fig. 4. The temperature dependence of (a) the order parameter (M), (b) the internal energy (H_I), (c) the magnetic susceptibility (χ), and (d) the specific heat (C, in units of k_B) of the four-dimensional Ising model computed with the three-bit demons for the lattice with $L = 4$, showing the effect of starting the simulation at a given total energy from different initial conditions for various quantities. The simulation lasts 9.6×10^5 sweeps.



(c)



(d)

Fig. 4. (*Continued*).

(9) The number of simulation steps, of course, determines the precision of the results at a given run at a given total energy. The values of the magnetization and the internal energy are not affected much by the number of simulation runs [Figs. 4(a) and 4(b)].[24] However, for increasing the precision of the magnetic susceptibility and the specific heat, the number of simulation runs must be increased [Figs. 4(c) and 4(d)].[24]