3. Methods of Analysis

(1) *Approximating the infinite lattice with a finite-size lattice*[35]

Near the critical temperature for the magnetic susceptibility of a finite-size lattice and below $T_c^{\chi}(L)$, a temperature interval where size effects are negligible is chosen and the data for the quantity of interest, usually the order parameter, is used for the infinite lattice. The analysis is carried out by using the formulas for the infinite lattice to get the values for its critical quantities.

The upper limit of the temperature interval of a finite-size lattice to be used as an approximation to the infinite lattice is determined by plotting the order parameters obtained for the lattices with successive L and finding the temperature where the curves begin to overlap, as T decreases (Fig. 5).[22] The lower limit for the temperature interval is taken as the value where the distortions on the curve(s) begin.

The simulations show that for a given L, the reduced temperature ϵ for the upper limit of the temperature interval decreases with increasing d. Thus, for a given L, the same ϵ can safely be used in larger dimensionalities.[22,35] The lower limit of the temperature interval is $\epsilon \simeq 0.09$ for all $2 \leq d \leq 8$.

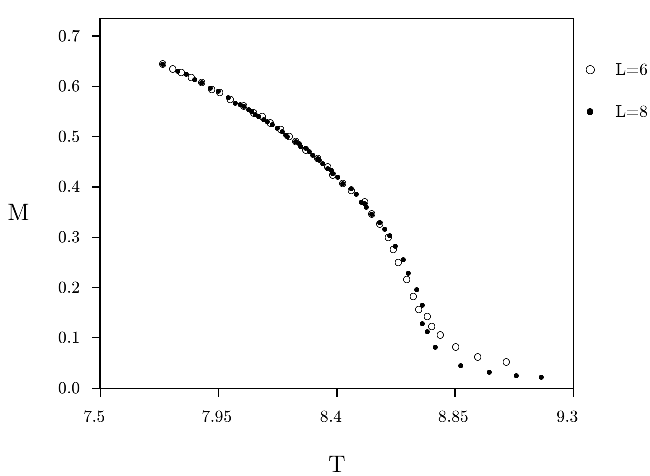

Fig. 5. The temperature dependence of the order parameter (M) of the five-dimensional Ising model computed with the four-bit demons for the lattices with $L = 6$ and 8. The simulation lasts 6×10^4 sweeps.

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(2) *Finite-size scaling theory*[36]

In $d = 2$ and 3 dimensions, the finite-size scaling theory gives the following scaling forms for the quantities of interest:

$$M = L^{-\beta/\nu} X(x) , \tag{4}$$

$$\chi = L^{\gamma/\nu} Y(x) , \tag{5}$$

$$C = L^{\alpha/\nu} Z(x) , \tag{6}$$

where $x = \epsilon L^{1/\nu}$, $\epsilon = \mid T - T_c \mid / T_c$ is the reduced temperature, and T_c is the critical temperature of the infinite lattice. The shape functions X, Y, and Z behave asymptotically as

$$X(x) = Bx^{\beta} , \tag{7}$$

$$Y(x) = Gx^{-\gamma} , \tag{8}$$

$$Z(x) = Ax^{-\alpha} . \tag{9}$$

Eqs. (4)–(6) take the following forms at $T = T_c$,

$$M \propto L^{-\beta/\nu} , \tag{10}$$

$$\chi \propto L^{\gamma/\nu} , \tag{11}$$

$$C \propto L^{\alpha/\nu} , \tag{12}$$

and at $T = T_c(L)$,

$$\chi_{\max} \propto L^{\gamma/\nu} , \tag{13}$$

$$C_{\max} \propto L^{\alpha/\nu} . \tag{14}$$

The finite-size scaling relation for $T_c(L)$ is

$$T_c - T_c(L) \propto L^{-1/\nu} . \tag{15}$$

The critical exponents α, β, γ, and ν are those of the infinite lattice.

In the upper critical dimension $d = 4$, the finite-size scaling relations used in the analysis are given below[37–39]:

The finite-size scaling relation for $T_c^C(L)$ is[37]

$$T_c^C(\infty) - T_c^C(L) \propto L^{-2} \log^{-1/6} L\,. \tag{16}$$

This relation is assumed to be valid also for $T_c^\chi(L)$.

At $T = T_c$,[39]

$$\chi(L) \propto L^2 \log^{1/2} L\,, \tag{17}$$

or $\chi(L) \propto L^{\gamma/\nu} \log^{1/2}(L)$. This relation is assumed to be valid also at $T = T_c^\chi(L)$.

At $T = T_c^C(L)$ and at $T = T_c$,[37,39]

$$C(L) \propto \log^{1/3} L\,, \tag{18}$$

or $C(L) \propto L^{\alpha/\nu} \log^{1/3}(L)$ with $\alpha = 0$.

In $d \geq 5$ dimensions, the finite-size scaling relations used in the analysis are given below[40,41]:

At $T = T_c$,

$$M \propto L^{-d/4} \tag{19}$$

and

$$\chi \propto L^{d/2}\,. \tag{20}$$

The finite-size scaling relation for $T_c(L)$ is

$$T_c - T_c(L) \propto L^{-d/2}\,. \tag{21}$$

(3) *Dynamic finite-size scaling hypothesis*[42]

The critical slowing down is determined by the linear (equilibrium) dynamical critical exponent which, in turn, is determined by the relaxation time $\tau(L)$. It has the following scaling form in $d = 2$ and 3 dimensions:

$$\tau = L^z f(x)\,. \tag{22}$$

For large x, $f(x)$ has the following asymptotic form:

$$f(x) = Cx^{-z\nu}\,. \tag{23}$$

Eq. (22) takes the following form at $T = T_c$ and $T = T_c(L)$:

$$\tau \propto L^z\,. \tag{24}$$

$\tau(L)$ are computed from the exponentially decaying part of the correlation function $\phi(t)$ which, in general, can be expressed as a linear combination of exponential functions,[43]

$$\phi(t) = \sum_i A_i e^{-t/\tau_i} \,. \tag{25}$$

The correlation function itself is computed according to the standard definition of correlation for two sets of data a_i and b_i[44]:

$$\phi(t) = \frac{N\sum_{i=1}^{N} a_i b_i - (\sum_{i=1}^{N} a_i)(\sum_{i=1}^{N} b_i)}{[N\sum_{i=1}^{N} a_i^2 - (\sum_{i=1}^{N} a_i)^2]^{1/2}[N\sum_{i=1}^{N} b_i^2 - (\sum_{i=1}^{N} b_i)^2]^{1/2}} \tag{26}$$

In the upper critical dimension $d = 4$, the finite-size scaling relation used in the analysis for the linear relaxation of the order parameter at $T = T_c$ and at $T = T_c^{\chi}(L)$ is assumed to have the same form as the magnetic susceptibility[21]:

$$\tau(L) \propto L^z \log^{1/2} L \,. \tag{27}$$

Another way to determine the dynamical exponent z is to make use of the nonlinear relaxation of the magnetization at T_c for very large lattices and very short times, less than L^z, such that the size effects are negligible[17,45–49]:

$$M \propto t^{-\beta/z\nu} \,. \tag{28}$$

4. Results and Discussion

4.1. $d = 2$

The two-dimensional Ising model is simulated with the nearest neighbor,[20,26,27] with the nearest neighbor and the next-nearest neighbor,[28] and with the nearest neighbor, the next-nearest neighbor and the four-spin interactions.[29] It is also simulated with a modified version of the Creutz cellular automaton which associates with each spin various number of two-bit demons.[30] The static[20,27–30] and dynamical critical exponents[26,30] are computed.

For the original case, including only the nearest-neighbor interaction and one demon for each spin,[19,20,26,27] the initial kinetic energy is given to the lattice via the second bits of the demons, such that the value of the initial kinetic energy for such a demon is 8 (in units of J). The simulations are carried out for the lattices with $30 \leq L \leq 120$, and the cellular automaton develops 50,000 time steps (25,000 sweeps). Analysis method 2 and 3 are used to determine the static,[20,27–30] and dynamical critical exponents,[26,30] and the critical temperature of the infinite lattice. The results are given in Table 1.