$\tau(L)$ are computed from the exponentially decaying part of the correlation function $\phi(t)$ which, in general, can be expressed as a linear combination of exponential functions,[43]

$$\phi(t) = \sum_i A_i e^{-t/\tau_i} \,. \tag{25}$$

The correlation function itself is computed according to the standard definition of correlation for two sets of data a_i and b_i[44]:

$$\phi(t) = \frac{N\sum_{i=1}^{N} a_i b_i - (\sum_{i=1}^{N} a_i)(\sum_{i=1}^{N} b_i)}{[N\sum_{i=1}^{N} a_i^2 - (\sum_{i=1}^{N} a_i)^2]^{1/2}[N\sum_{i=1}^{N} b_i^2 - (\sum_{i=1}^{N} b_i)^2]^{1/2}} \tag{26}$$

In the upper critical dimension $d = 4$, the finite-size scaling relation used in the analysis for the linear relaxation of the order parameter at $T = T_c$ and at $T = T_c^{\chi}(L)$ is assumed to have the same form as the magnetic susceptibility[21]:

$$\tau(L) \propto L^z \log^{1/2} L \,. \tag{27}$$

Another way to determine the dynamical exponent z is to make use of the nonlinear relaxation of the magnetization at T_c for very large lattices and very short times, less than L^z, such that the size effects are negligible[17,45–49]:

$$M \propto t^{-\beta/z\nu} \,. \tag{28}$$

4. Results and Discussion

4.1. $d = 2$

The two-dimensional Ising model is simulated with the nearest neighbor,[20,26,27] with the nearest neighbor and the next-nearest neighbor,[28] and with the nearest neighbor, the next-nearest neighbor and the four-spin interactions.[29] It is also simulated with a modified version of the Creutz cellular automaton which associates with each spin various number of two-bit demons.[30] The static[20,27–30] and dynamical critical exponents[26,30] are computed.

For the original case, including only the nearest-neighbor interaction and one demon for each spin,[19,20,26,27] the initial kinetic energy is given to the lattice via the second bits of the demons, such that the value of the initial kinetic energy for such a demon is 8 (in units of J). The simulations are carried out for the lattices with $30 \leq L \leq 120$, and the cellular automaton develops 50,000 time steps (25,000 sweeps). Analysis method 2 and 3 are used to determine the static,[20,27–30] and dynamical critical exponents,[26,30] and the critical temperature of the infinite lattice. The results are given in Table 1.

Table 1. The values of the critical exponents and the critical temperature for the two-dimensional Ising model computed on the Creutz cellular automaton.[20,26,27] (The values listed for Ref. 20 and Ref. 27 are the averages of independent estimations given there.)

Quantity	Ref. 20	Ref. 27	Ref. 26	References
T_c	2.263	—	—	$2\log_e^{-1}(1+2^{1/2}) = 2.26918\ldots$
α	0.05	—	—	0(log)
β	0.125	—	—	1/8
γ	1.81	—	—	7/4
ν	—	1.00	—	1
η	—	0.24	—	1/4
z_M	—	—	2.20	2.16 [11], 2.17 [49], 2.18 [50], 2.165 [51], 2.183 [52], 2.19 [53], 2.24 [54], 2.27, 2.16 [55], 2.16(3) [56], 2.1665(12) [57]
z_E	—	—	2.20	2.2 [58]

The simulations in Refs. 28–30 reproduce the above values of the critical exponents and the critical temperature, within the error limits. The critical values are in agreement with the analytical and Monte Carlo results.

The simulation of the Ising model on the modified version of the Creutz cellular automaton[30] shows that the specific heat curve approaches, in the limit, to that of the canonical ensemble as the number of demons per spin increases. The curves for the magnetic susceptibility and the order parameter are not affected. This behaviour is observed also in the simulations of the Ising model in higher dimensions, as the number of bits (the number of energy levels) of the demon increases.[22,31,32]

4.2. $d = 3$

The three-dimensional Ising model is simulated with the two- and three-bit demons[31] (in Ref. 25 with the four-bit demons also). The initial kinetic energy is given to the lattice via the first and second bits of the demon such that the value of the initial kinetic energy for such a demon is 12 (in units of J).

Simulations with the three-bit demons result in sharper and higher peaks for the magnetic susceptibility and the specific heat compared to those obtained with the two-bit demons. Therefore, the data obtained from the simulations with the three-bit demons are used in the analysis.

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Table 2. The values of the critical exponents and the critical temperature for the three-dimensional Ising model computed on the Creutz cellular automaton.[25,31] (The values listed for Ref. 31 are the averages of independent estimations given there.)

Quantity	Ref. 31	Ref. 25	References
T_c	4.53	—	4.51154 [59], 4.511524(20) [60], 4.51142(5) [61], 4.511528(6) [62], 4.51158(8) [63]
α	0.07	—	0.110(2) [60], 0.105(7) [64], 0.105(10) [65]
β	0.30	—	0.3267(10) [60], 0.3258(44) [61], 0.3270(15) [66]
γ	1.25	—	1.237(2) [60], 1.2390(71) [61], 1.2395(4) [64], 1.237(2) [65], 1.2390(25) [66]
z_M	—	2.1	2.05 [49], 2.04 [50], 2.09(6) [56], 2.04(3) [67], 2.02 [68], 2.06(2) [69], 2.09(2) [70]

The cellular automaton develops 60,000 time steps (30,000 sweeps) in computing the static,[31] and 3.6×10^6 time steps in computing the dynamical critical exponents.[25]

Analysis method 2 and 3 are used to determine the critical exponents and the critical temperature of the infinite lattice, by using the lattices with $6 \leq L \leq 14$; for larger L, χ and C are underestimated at and very near the critical temperatures. The results are listed in Table 2. They are in agreement with the Monte Carlo results within the error limits.

4.3. $d = 4$

The four-dimensional Ising model is simulated with the three-bit demons.[21,23] In Ref. 21, the initial kinetic energy is given to the lattice via the third bit of the demons such that the value of the initial kinetic energy for such a demon is 16 (in units of J). In Ref. 23, the initial kinetic energy is given to the lattice via the first and the third bits of the demons such that the value of the initial kinetic energy for such a demon is 20, which makes the simulations at total energies larger than the maximum values in Ref. 21 possible. In Ref. 21, the cellular automaton develops 60,000 time steps (30,000 sweeps), and in Ref. 23 it develops 9.6×10^5 sweeps for the lattices with $L \leq 10$, and 3.6×10^5 sweeps for the larger lattices.

Analysis methods 1, 2 and 3 are used to determine the critical exponents and the critical temperature of the infinite lattice, by using the simulations

Table 3. The values of the critical exponents and the critical temperature for the four-dimensional Ising model computed on the Creutz cellular automaton.[21,23]

Quantity	Ref. 21	Ref. 23	References
T_c	6.68	6.680	6.680 [38,39], 6.6802 [48], 6.6803 [48], 6.682 [72]
α	-0.04	0.003	0
β	0.50	—	1/2
γ	1.02	1.00	1
z_M	2.02	—	2.00 [48,50]

on the finite-size lattices with $4 \leq L \leq 16$. The results are listed in Table 3. They are in agreement with the renormalization group predictions,[37,71] the series expansion,[72] and the Monte Carlo results.[35,38,39,48]

4.4. $d = 5$

The five-dimensional Ising model is simulated with the three- to five-bit demons.[22] The initial kinetic energy is given to the lattice via the second and the third bits of the demon such that the value of the initial kinetic energy for such a demon is 24 (in units of J). The cellular automaton develops 60,000 time steps (30,000 sweeps) outside just about the critical temperatures of each lattice, where 120,000 time steps are used.

Analysis method 1 is used to determine the critical temperature of the infinite lattice by using the data for the order parameter of the lattice with $L = 8$. In order to determine the upper limit of the temperature interval of this lattice, which will be used as an approximation to the infinite lattice, the order parameters obtained with the four-bit demons for the lattices with $L = 6$ and 8 are plotted (Fig. 5). The temperature where the curves begin to overlap, which happens at $T = 8.51$ (Fig. 5), is taken as the upper limit of the temperature interval. The lower limit for the temperature interval is taken as the value where the distortions on the curve(s) begin. This happens at $T = 8.07$ for the lattice with $L = 8$. Therefore, both the lattices with $L = 6$ and 8 approximate the infinite lattice within the interval $8.07 \leq T \leq 8.51$. The data for $L = 8$ are fit to the expression for the order parameter of the infinite lattice in the form of a power law with correction[73]:

$$M = B\epsilon^{\beta}(1 + b\epsilon^{\Delta_M} + \cdots)\,, \qquad T < T_c\,, \qquad T \to T_c\,, \tag{29}$$

Table 4. The values of the critical amplitude and the critical temperature of the infinite lattice for the five-dimensional Ising model, obtained by fitting the data for the order parameter of the finite-size lattice to the power law with correction within the temperature interval where it approximates the infinite lattice.

L	Bits	Interval	B	T_c	References
8	3	$8.07 \leq T \leq 8.51$	2.1	8.80	8.7769(12) [45], 8.77886(77) [45, 47]
	4		2.1	8.79	8.780(10) [74], 8.7812(23) [75]
	5		2.2	8.79	8.778475(31) [76]

where B is the critical amplitude, β is the leading critical exponent, b and Δ_M are the correction amplitude and exponent, respectively. Meanwhile, the critical temperature of the infinite lattice T_c, is varied within the interval $8.73 \leq T_c \leq 8.84$ in order to determine that value satisfying the renormalization group prediction of $\beta = 1/2$ within the error limits ± 0.01. The results are listed in Table 4.

The values of the infinite-lattice critical temperature (with the errors ± 0.02) and the critical amplitude for the order parameter (with the errors ± 0.1) are in agreement with the recent results within the error limits.[45–47,74–76]

Presently more precise simulations are carried out on the Creutz cellular automaton for the five-dimensional Ising model for the three- and five-bit cases just about the critical temperature of each lattice and at the infinite-lattice critical temperature, for the finite-size lattices with $4 \leq L \leq 12$.[33] The cellular automaton develops 9.6×10^5 ($L = 4$), 3.6×10^5 ($L = 6$), and 9.6×10^4 ($8 \leq L \leq 12$) sweeps for each run, with 7 runs for each total energy. The finite-size scaling relations for χ and M at T_c[41] are tested. The slopes of their $\log - \log$ plots against L give 2.50 and 1.26 ($6 \leq L \leq 12$) at $T_c = 8.779$, in very good agreement with the theoretical predictions, $d/2$ (5/2) and $d/4$ (5/4), respectively. The Monte Carlo simulations[77] also confirm these and other predictions of the theory.

4.5. $d = 6$

The six-dimensional Ising model is simulated with the three- to five-bit demons.[22] In simulating with the three-bit demons, the initial kinetic energy is given to the lattice via the first, the second, and the third bits, such that the value of the initial kinetic energy for such a demon is 28 (in units of J). In simulating with the four- and five-bit demons, the initial kinetic energy is

Table 5. The values of the critical amplitude and the critical temperature of the infinite lattice for the six-dimensional Ising model, obtained by fitting the data for the order parameter of the finite-size lattice to the power law with correction within the temperature interval where it approximates the infinite lattice.

L	Bits	Interval	B	T_c	References
6	3	$9.97 \le T \le 10.52$	2.0	10.84	10.83482(35) [46]
	4		2.0	10.84	
	5		2.0	10.84	

given to the lattice via the first and the fourth bits, such that the value of the initial kinetic energy for such a demon is 36. The cellular automaton develops 60,000 time steps (30,000 sweeps) outside just about the critical temperatures of each lattice, where 120,000 time steps are used.

Analysis method 1 is used to determine the critical temperature of the infinite lattice by using the data for the order parameter of the lattice with $L = 6$. The lower limit for the temperature interval is taken as the value where the distortions on the curve(s) begin. This happens at $T = 9.97$ for the lattice with $L = 6$.[22] Simulations show that for a given L, $\epsilon(L)$ for the upper limit of the temperature interval decreases with increasing d. Thus, for a given L, the same $\epsilon(L)$ can safely be used in larger dimensionalities.[22,35] For $L = 6$ in $d = 5$ dimensions, $\epsilon = 0.031$ for the upper limit of the temperature interval. This, together with the known value of $T_c = 10.84$,[46] gives $T = 10.52$ for the upper limit of the temperature interval for $L = 6$ in $d = 6$ dimensions. Thus, the order parameter data within the interval $9.97 \le T \le 10.52$ are fit to the expression in Eq. (29), while T_c is varied within the interval $10.78 \le T_c \le 10.89$. The results are listed in Table 5.

The values of the infinite-lattice critical temperature (with the errors ± 0.02) and the critical amplitude for the order parameter (with the errors ± 0.1) are in agreement with the recent results within the error limits.[46]

4.6. $d = 7$

The seven-dimensional Ising model is simulated with the three- to five-bit demons.[22] The initial kinetic energies are given to the lattice as in the case of $d = 6$ dimensions. The cellular automaton develops 60,000 time steps (30,000 sweeps) outside just about the critical temperatures of each lattice, where 120,000 time steps are used.

Table 6. The values of the critical amplitude and the critical temperature of the infinite lattice for the seven-dimensional Ising model, obtained by fitting the data for the order parameter of the finite-size lattice to the power law with correction within the temperature interval where it approximates the infinite lattice.

L	Bits	Interval	B	T_c	References
4	3	$11.84 \leq T \leq 12.43$	1.9	12.87	12.86902(33) [46]
	4		2.0	12.87	
	5		1.9	12.86	

Analysis method 1 is used to determine the critical temperature of the infinite lattice by using the data for the order parameter of the lattice with $L = 4$. The lower limit for the temperature interval is taken as the value where the distortions on the curve(s) begin. This happens at $T = 11.84$ for the lattice with $L = 4$.[22] In $d = 6$ dimensions, the curves for the order parameter of the lattices with $L = 4$ and 6 obtained from the simulations with the four-bit demons begin to overlap at $T = 10.46$.[22] Therefore, for both $L = 4$ and 6, the size effects are negligible for $T \leq 10.46$. In terms of the reduced temperature, the upper limit of the temperature interval is $\epsilon(L) = 0.035$ for $L = 4$. This value of $\epsilon(L)$ can be taken as the upper limit of the temperature interval for $L = 4$ in $d = 7$ dimensions.[22,35] This, together with the known value of $T_c = 12.87$,[46] gives $T = 12.43$ for the upper limit of the temperature interval for $L = 4$ in $d = 7$ dimensions. Thus, the lattice with $L = 4$ approximates the infinite lattice in $d = 7$ dimensions within the interval $11.84 \leq T \leq 12.43$. The order parameter data are fit to the expression in Eq. (29), while T_c is varied within the interval $12.80 \leq T_c \leq 12.95$. The results are listed in Table 6.

The values of the infinite-lattice critical temperature (with the errors ± 0.02) and the critical amplitude for the order parameter (with the errors ± 0.1) are in agreement with the recent results within the error limits.[46]

4.7. $d = 8$

The eight-dimensional Ising model is simulated with the four-bit demons.[34] The initial kinetic energy is given to the lattice via the second and the fourth bits of the demon such that the value of the initial kinetic energy for such a demon is 40 (in units of J). The cellular automaton develops 60,000 time steps (30,000 sweeps).

Analysis method 1 is used to determine the critical temperature of the infinite lattice by using the data for the order parameter of the lattice with $L = 4$. The lower limit for the temperature interval is taken as the value where the distortions on the curve(s) begin. This happens at $T = 13.664$ for the lattice with $L = 4$.[34] The data for the order parameter within the interval $13.664 \leq T \leq 14.370$ are fit to the power law with correction in Eq. (29), using the criterion $0.49 \leq \beta \leq 0.51$. In doing this, T_c is varied within the interval $14.820 \leq T_c \leq 15.000$. The results for the critical temperature and the critical amplitude of the infinite lattice are $T_c = 14.898 \pm 0.010$ and $B = 1.9 \pm 0.1$, respectively. At this value of the critical temperature, $\epsilon \geq 0.035$ for the upper limit of the temperature interval is satisfied, implying that the size-effects in these data are negligibly small.[22,35] The value of the critical temperature is in agreement with the $1/d$-expansion result of $T_c = 14.892$[78] and the recent value of $T_c = 14.8911 \pm 0.0002$.[79]

5. Conclusion

The review of the results obtained from the simulation of the Ising model on the Creutz cellular automaton leads to the following conclusions: it is a very good approximation to the Ising model above a limiting temperature. This temperature region includes a sufficiently large interval below the critical temperature of the magnetic susceptibility where the infinite lattice can be approximated. The degree of approximation is revealed by the simulations with increased accuracy. Presently for the critical temperatures, the third digit after the decimal point, and for the critical exponents, the second digit have been reached in accuracy, and within this accuracy the theoretical predictions and the Monte Carlo results are reproduced. This review shows that the Creutz cellular automaton can be used as an alternative research tool for Ising model investigations, and it is now at the stage of testing the predictions of the theory, rather than being tested.

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