$$\alpha(p) = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & 2 < p \le 4\,, \\ \frac{1}{2} - \frac{3}{2p} & 4 < p \le \infty\,. \end{cases} \tag{1.19}$$

In Ref. 10 inequality (1.18) was obtained by an adaptation of Córdoba's proof of the Carleson–Sjölin theorem.[3] Note that (1.19) represents a gain of a fixed number of derivatives over the fixed time estimate (1.15), e.g., in case $p = 4$ of a gain $< 1/8$. We therefore conclude from the previous discussion that Bourgain's circular maximal theorem is a consequence of (1.18). A conjecture of Sogge,[15] known as the sharp local smoothing conjecture, maintains that (1.18) should hold with $2 < p \le 4$ and $\alpha(p) = 0$. It was shown in Ref. 15 that this is the best possible and that it would imply the Carleson–Sjölin theorem. Moreover, it is easy to see that the optimal $L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$ bounds for the circular maximal function would follow from Sogge's conjecture. Indeed, applying Sobolev imbedding in t as above, that conjecture would imply

$$\|\bar{\mathcal{M}} f\|_{L^4(\mathbb{R}^2)} \le C\|f\|_{L^4_{-1/4+\varepsilon}(\mathbb{R}^2)}\,.$$

In conclusion we would like to mention that Schlag and Sogge[13] have recently established the following local smoothing estimate

$$\int_1^2 \int_{\mathbb{R}^2} |Ff(x,t)|^5 \, dxdt \le C_\varepsilon \|f\|^5_{L^{5/2}_{3/10+\varepsilon}}\,. \tag{1.20}$$

Firstly, (1.20) would again follow from the sharp local smoothing conjecture by interpolating with the appropriate $L^1_x \to L^\infty_{x,t}$ bound. Secondly, since $3/10 = 1/2 - 1/5$, Sobolev imbedding with $1/5 + \varepsilon$ derivatives in t shows that Theorem 1.1.2 follows from (1.20). The proof of (1.20) is based in part on the Klainerman–Machedon estimate[7] for the wave equation. However, we shall not discuss the proof here, since the emphasis in this thesis lies on the combinatorial method of Kolasa–Wolff rather than on the Fourier integral operator methods. Reference 13 also contains variable coefficient extensions of Theorem 1.1.2 as well as sharp $L^p \to L^q$ bounds for spherical maximal operators in higher dimensions.

1.3. *Related questions from geometric measure theory*

Suppose E and F are subsets of the plane with E closed and F measurable. Assume that for every $x \in F$ there is an $r_x \in (1,2)$ such that

$$\mathcal{H}^1(C(x,r_x) \cap E) > 0 \qquad \text{for all } x \in F\,,$$

where $\mathcal{H}^1$ is linear Hausdorff measure (for properties of Hausdorff measure, see Falconer's book.[5] Chap. 1 of dimension in this section will always mean Hausdorff dimension). In this section we shall always assume that E and F are as above. The following result is due to Marstrand[9] and Bourgain.[1] We will denote Lebesgue measure by $|\cdot|$.

Theorem 1.3.1. *If* $|F| > 0$, *then* $|E| > 0$. *In other words, the union of a family of circles has positive (planar) measure if their centers form a set of positive measures.*

Proof. This is a simple consequence of Bourgain's circular maximal theorem. Indeed, assume that $U \in \mathbb{R}^2$ is open. Then there is an increasing sequence of non-negative functions $f_n \in C_c(\mathbb{R}^2)$ so that $\sup_n f_n = \chi_U$. Fix a $p \in (2, \infty)$. By monotone convergence and Theorem 1.1.1

$$\|\bar{\mathcal{M}}\chi_U\|_p \le C\|\chi_U\|_p = C|U|^{1/p} .$$

In case $|E| = 0$ there would be a decreasing sequence of open sets $U_n \supset E$ such that $\bigcap_n U_n = E$ and $|U_n| \to 0$. We could then conclude from the previous inequality that

$$\|\bar{\mathcal{M}}\chi_E\|_p = 0 .$$

However, this contradicts $|F| > 0$. ☐

One might ask whether the conclusion of Theorem 1.3.1 will still hold under the assumption that $\dim(F) > c_0$ for some $c_0 < 2$. The following result of Talagrand[21] shows that c_0 has to be at least one.

Proposition 1.3.1. *There exist* E *and* F *such that* $\mathcal{H}^1(F) > 0$ *but* $|E| = 0$.

On the other hand, Wolff[22] has shown recently that for F as in the proposition, E "barely fails to have positive measure". More precisely he showed

Theorem 1.3.2. *If* $\mathcal{H}^1(F) > 0$, *then* $\dim(E) = 2$.

His proof is based in part on an argument from combinatorial geometry that was developed in Ref. 4 to obtain bounds on the number of incidences between n spheres and m points in $\mathbb{R}^3$.

We do not know whether it is possible to deduce $c_0 = 1$ from Ref. 22. Rather, we will show that $c_0 = 1$ would be a consequence of the sharp local smoothing conjecture from the previous section. This connection, which relies on the theory of capacities, was brought to our attention by Thomas Wolff. For the definition of capacity as well as the connection between Hausdorff measure and capacity we refer the reader to the Appendix.

Proposition 1.3.2. *Given that the sharp local smoothing conjecture is true, suppose that* $\dim(F) > 1$. *Then* $|E| > 0$.

Proof. Firstly, we may assume that E is compact and that

$$\mathcal{H}^1(E \cap C(x, r_x)) \geq \gamma > 0$$

for all $x \in F$ with γ fixed (since $\mathcal{H}^1$ is a regular outer measure, see Ref. 5). Since $\dim(F) > 1$ there is an $\varepsilon > 0$ such that $\mathcal{H}^{1+\varepsilon}(F) > 0$. Fix ε to be that number. Assume that the proposition fails. Then there is a sequence f_j of non-negative functions in $C_0^\infty(\mathbb{R}^2)$ such that $f_j = 1$ on a neighborhood of E and $\|f_j\|_4 \to 0$ as $j \to \infty$. Pick a cutoff function $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta = 1$ on $(1, 2)$. Define

$$u_j(x, r) = (d\sigma_r * f_j)(x)\eta(r)\,.$$

Then $u_j > \gamma/2$ on some neighborhood of $F' = \{(x, r_x) : x \in F\}$. If the sharp local smoothing conjecture is correct, then

$$\|u_j\|_{L^4_{1/2-\varepsilon/8}(\mathbb{R}^3)} \leq C_\varepsilon \|f_j\|_{L^4(\mathbb{R}^2)}\,.$$

Therefore, by the definition of capacity (see the Appendix)

$$C_{4,1/2-\varepsilon/8}(F') \leq C_\varepsilon \|f_j\|_4\,.$$

Passing to the limit $j \to \infty$, this would imply that

$$C_{4,1/2-\varepsilon/8}(F') = 0\,. \tag{1.21}$$

However, by the proposition in the Appendix we conclude from (1.21) that

$$0 = \mathcal{H}^{3-4(1/2-\varepsilon/8)+\varepsilon/2}(F') = \mathcal{H}^{1+\varepsilon}(F') \geq \mathcal{H}^{1+\varepsilon}(F)$$

which contradicts the choice of ε. □

TOPICS IN ANALYSIS AND ITS APPLICATIONS - Selected Theses