Introduction

The five selected theses in analysis, by A. Gilbert, N. Saito, W. Schlag, T. Tao and C. Thiele, cover a broad spectrum of modern Harmonic Analysis from Littlewood–Paley theory (wavelets) to subtle interactions of geometry and Fourier oscillations. The common theme through the theses involves intricate local Fourier (or multiscale) decompositions of functions and operators to account for cumulative properties involving size or structure.

Anna Gilbert's thesis addresses the question of homogenization for partial differential equations and transitions from fine scale to course scales.

The purpose is to provide effective equations to describe the solution of a PDE in mixed media. This is achieved by proving that solutions of governing PDE in unmixed media converge weakly to solutions of some effective equation as the medium is randomly mixed.

By setting the problem up as a sequence of transitions from solutions of equations on a fine (micro) scale and by deriving equations for the averages of the solution on a coarser scale, A. Gilbert provides a general method for direct numerical analysis for the transitions to macroscopic scales. She manages, for various examples, to compute analytically the effective equations with surprising results.

The main point is that effective transitions from a microscopic regime to a coarse regime can be tracked without having to assume that we have random mixing. It is enough to describe transitions between neighboring scales.

The use of wavelets in this work provides the algebraic structure needed to convert Littlewood–Paley theory into a systematic numerical tool, by providing a detailed account of transitions from high frequencies in the fine structured equations to low frequencies in the effective coarse structure.

Terence Tao in his thesis develops tools for the analysis of various classes of singular integral operators. In the first part he shows that the spherical Fourier restriction property of Thomas–Stein implies a corresponding weak type estimate for Bochner–Riesz multipliers. (This result is then generalized for uniformly elliptic first-order pseudo-differential operators.) Here again the important contribution is in the method of decomposition of this oscillatory operator, which allows for a precise control of its action on the corresponding Calderón–Zygmund decomposition of the function.

In the next part of his thesis, Tao shows that the numerically natural way for truncating a wavelet expansion is a.e. convergent. (The same statement is false by Fourier Series as shown by T. Korner.)

In the last part he generalizes a weak type estimate of A. Seeger for even singular integrals with Ω in $L \log L$ to the homogeneous group setting. Here again the crucial issue is to obtain a good decomposition of the operator.

N. Saito's thesis is concerned with the decomposition and analysis of functions for extraction of "structures and features" useful for discrimination and regression. Here again the main point is to represent a function as a combination of local Fourier series, where the local expansion is chosen so that the large coefficients will be useful for characterizing the "geometry" of the function: This can be achieved by finding an orthogonal local sine basis which minimizes the description length of the function. Similarly a local basis can be chosen to maximize discrimination between two classes of signals (functions).

This thesis in applied mathematics develops a variety of analysis tools which are then applied to the analysis of acoustic ground signatures for identification and discrimination.

The time frequency representations used here to decompose functions are also the fundamental tool in Thiele's thesis concerning the bilinear Hilbert transform of Calderón.

Wilhelm Schlage addresses the problem of estimating the maximal circular averages operator by geometric combinatorial methods.

Ever since E. Stein discovered, by using the Fourier transform, that the maximal spherical operator is bounded on some L^p in $\mathbb{R}^2$ for $n \geq 3$ and Bourgain proved the result in $\mathbb{R}^2$, there have been numerous attempts to prove these results by purely geometric means in which careful combinatorial coverings by annuli lead to geometric measure theoretic estimates.

Schlage extends the methods of Bourgain, Kolasa–Wolff to extend the known range of (L^p, L^q) estimates. Again the main point is the subtle interaction between the geometry of circles and Fourier analysis estimates. The maximal operator is a generalized Fourier integral operator requiring considerable geometric insight to obtain the appropriate decomposition of the operator.

C. Thiele in his work starts by recasting the a.e. convergence for Walsh series in terms of decomposition theorems in the Walsh multiscale phase plane.

This work builds on the insight of Carleson, Hunt, Fefferman, and serves as a warmup to the attack on the Walsh model for the bilinear Hilbert transform, which together with Michael Lacey, leads to the proof of Calderón's conjecture on the boundedness of the bilinear Hilbert transform. (For which they were awarded the Salem Prize.)

Here again we see the same main ingredients. The functions are broken up into a multiscale Fourier series in a way that controls and simplifies the bilinear (and trilinear) interactions. These decompositions are obtained by peeling off layers of large Fourier coefficients at different scales so as to control the operator.

The methodology developed in this thesis and subsequent work with M. Lacey have converted Carleson's "tour de force" in proving the a.e. convergence of Fourier series into a general tool of Harmonic analysis.

It is quite clear that the main contribution of all five Ph.D's is in providing deep organizational insight to complex interactions between localized Fourier components of the operators or functions being studied. The theorems and results are important, but it is the various methods of decomposition which are the enduring life blood of analysis. These tools have a much broader range of applicability, from linear and nonlinear partial differential equations to fast numerical algorithms, and are bound to have a lasting impact.

These theses are beautifully written and are either self-contained or well referenced. They provide a valuable advanced textbook for a "topics" course in Harmonic analysis and its applications.