Chapter 1

Introduction and Background

1.1 Interaction of Physics and Mathematics

The history of physics, especially over the last several hundred years, is replete with examples of the cross fertilization between Physics and (otherwise) Pure Mathematics. Examples and anecdotes abound. In the context of this book, the development of geometry (and its generalizations) as mathematics and as physics is especially noteworthy. In the 1700's mathematicians began in earnest the questioning of the minimal structure of Euclid's axioms, in particular, the necessity or not of including the postulate pertaining to parallel lines. Max Jammer [Jammer (1960)] summarizes this history very well. Underlying the mathematical discussion is the additional question of whether or not the axioms are physical or mathematical in nature. Of course, today, we are quite comfortable with the separation of pure mathematics from physics, but this has not always been so clear. Thus, for example, as described in [Jammer (1960)], Gauss actually performed a physical experiment with surveying equipment to determine if the sum of the angles in a triangle is indeed π, as it should be in flat, Euclidean, geometry. He bounced light off of three mirrors constituting the three vertices of a triangle. Of course, with the technology available to him at the time, such an experiment could be done with only crude accuracy, but it presaged a whole set of experiments on the behavior of light rays undertaken over the last thirty years or so within the solar system.

Such work by Gauss, Riemann, Lobachevski and others on the apparently very abstract and non-physical subject of "non-Euclidean" geometry was precisely what was needed to provide the foundation for Einstein's theory of General Relativity, in which gravity is described in terms of the geometric properties of spacetime. The path by which Einstein was led to consider what must have appeared to him to be very abstruse and abstract mathematics as a possible tool for physics has recently been reviewed in the various volumes celebrating the centennial of his birth. A very nice summary is provided by Norton[Norton (1992)].

Later investigations of Einstein's theory led to the natural introduction of non-trivial topology in addition to geometry. In the meantime, the parallel development of quantum theory and quantum field theory has led to the introduction into physics of branches of mathematics such as function theory, Hilbert spaces, bundle theory, moduli space structures, etc. In fact, the second half of the twentieth century has seen a virtual explosion of applications of various branches of mathematics, many of which were considered to be of only abstract interest, to physics. Conversely, in many cases the direction of "applicability" has been reversed, some of which will be touched on in this book. Questions of interest in theoretical physics have turned out to have value in the pursuit of "pure" mathematics.

In summary, the rich interplay between physics and mathematics is obvious to contemporary workers. Certainly, there is no theorem that says "Good mathematics makes good physics," but certainly there is strong anecdotal evidence that this has been true in many important situations. The purpose of this book is to introduce to physicists some recent exciting discoveries in pure mathematics that prove the existence of non-trivial structures on spacetime models which have always been assumed by physicists, and probably most mathematicians, to be trivial only. Can these new structures have physical significance?

The relevant mathematical arena is "*differential topology*," a very descriptive name since it is concerned with global (topology) smoothness (differential) questions. Since almost all current physical theories make use of calculus at some level, the notion of differentiation on any spacetime model is certainly essential to physics. Furthermore, since the early days of relativity, the importance of global features of spacetime, i.e., topology, has been apparent. Recently, of course, topological features of various theoretical models have been important in a much wider class of theoretical constructs such as quantum field theory and attempts to quantize general relativity. The basic question motivating the studies in this book is whether or not there is any non-trivial relationship between the purely *local* nature of differentiation and the *global* nature of topology. The answer, since the pioneering work of Donaldson, Freedman, et al., is a resounding "yes," opening the door to obvious re-investigations of some fundamental assumptions of theoretical physics.

Before beginning the detailed study of the mathematics surrounding contemporary differential topology, let us recall briefly the roles that spacetime models play in physics, and what is required of them in this process. The history and philosophy of this subject is rich and much too involved and extensive for us to consider here. See the book of Jammer[Jammer (1960)] for one overview. Also, the history of general relativity is obviously intimately related to the development of spacetime structures, so the studies generated by the anniversary of Einstein's birth provide a more recent look at this subject. It is clear that spacetime models serve at least two roles:

(1) A spacetime model carries structures such as topology, smoothness, and geometry. To some extent or another, these features seem to have real, observable consequences[1].
(2) A spacetime model serves only as a computational "scratch pad," on which theories are expressed, calculations done, and experimental predictions made. Apart from this purpose, the model has no direct physical significance.

It is important to keep in mind these distinct roles. Someone trying to "understand" the physics of spacetime has an entirely different set of standards than one, perhaps as a worker in quantum field theory, who merely regards spacetime as a necessity for expressing field equations, perturbation theory, path integrals, etc. To the former, the innate difficulties in trying to give operational significance to spacetime in the light of quantum uncertainty principles for measurements are of profound importance. To the latter, these issues are peripheral at best and spacetime exists only as a platform for calculating integrals. Of course, most physicists at one time or another in their research probably find themselves taking each of the positions. In this book, of course, we clearly are prejudiced toward position (1).

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[1] The observability of these structures and thus their operational significance continues to be a deep unresolved issue. See, e.g., [Brans (1980)],[Brans (1999)]

1.2 Manifolds: Smoothness and Other Structures

Consider the following sequence:

$$\begin{tikzcd}
\dots \arrow[d, "?"] \\
\textbf{Point Set} \arrow[d, "\mathcal{T}"] \\
\textbf{Topological Manifold} \arrow[d, "\mathcal{S}"] \\
\textbf{Smooth Manifold} \arrow[d, "\mathcal{G}"] \\
\textbf{Geometric Manifold} \arrow[d, "\mathcal{B}"] \\
\textbf{Bundle} \arrow[d, "?"] \\
\dots
\end{tikzcd} \tag{1.1}$$

representing what seems to be the minimal set of necessary component assumptions that go into the underlying spacetime models of almost all physical theories.

At the lowest level some sort of *point set* structure seems to be required. That is, except for a brave few who attempt to replace point-set structures by something "derived" from more basic quantum structures the idea that spacetime is a point set seems to be universally, if generally tacitly, assumed. Historically the quantification of geometry led to the identification of this point set with Euclidean space, $\mathbb{R}^n_{\mathrm{PS}}$, whose points are the ordered sets of n real numbers. At its foundation, physics has not strayed too far from this point set model.

First, the transition $\mathcal{T}$: With the development of topology over the last 100 years, it has become obvious that more than the identity of points in a set is of significance. In fact, the notion of *limit* defining a *topology* is indispensable to any reasonable spacetime model. In modern usage, a *topology* can be defined in terms of *open sets*, or their generators, *neighborhoods.* Historically the "default" topology for numerical point sets such as $\mathbb{R}^n_{\mathrm{PS}}$ has generally been that defined by the real numbers which have provided the point set structure, the *Euclidean metric topology.* Thus, in the transition $\mathcal{T}$ in 1.1 from $\mathbb{R}^n_{\mathrm{PS}}$ to $\mathbb{R}^n_{\mathrm{TOP}}$, the latter uses neighborhoods defined by the balls generated by the real numbers chosen to define the point set[2]. But current mathematics and physics make use of a generalization of pure global Euclidean topology while still maintaining this structure locally. This leads to the notion of a *topological manifold* as a point set with topology which is *locally* Euclidean. That is, each point has a neighborhood homeomorphic to $\mathbb{R}^n_{\mathrm{TOP}}$. This latter, tautologically topological manifold, is the one most often used by physicists outside of general relativity, with little thought to alternatives.

Next, $\mathcal{S}$: Since physics uses fields and field equations, we need the machinery to perform *differentiation.* This is provided by a definition of how to do calculus locally, a "*smoothness (or differentiable) structure.*" The definition and study of such structures, *in the large*, will be the main preoccupation of this book. At its most basic, we need to define which (real-valued) functions on $\mathbb{R}^n_{\mathrm{TOP}}$ are to be regarded as "smooth," or equivalently "differentiable," and then decide how to construct the derivative. For simplicity, we restrict the notion of differentiability to "infinitely differentiable," or C^∞ in the usual notation. There are several routes to defining smoothness structures, the transition $\mathcal{S}$ in 1.1, and much of this book will be dedicated to this topic.

Current physics is based on the tacit assumption that this transition, $\mathcal{S}$, is trivial, that there is some *natural, standard smoothness* structure on $\mathbb{R}^n_{\mathrm{TOP}}$ given by the *topological* coordinates, x^i. So, in what seems to be a natural, minimal, and harmless (but in reality not) process, let us define the class of smooth functions on this smooth structure, $\mathcal{F}_0(\mathbb{R}^n_{\mathrm{TOP}})$, where the "0" means the standard structure, to be those real valued functions of these topological coordinates, $f(x^i)$, which are C^∞ differentiable in the usual real analysis sense. This defines $\mathbb{R}^n_{\mathrm{DIFF},0}$ as a candidate for "the" standard smooth Euclidean manifold.

Of course, it soon becomes clear that these naive procedures may not be well defined because of built-in assumptions that might not be obvious to those concerned only with physical applications. In particular, within the topological category we are really only concerned with equivalence classes under homeomorphisms. Such topological changes to $\mathbb{R}^n_{\mathrm{TOP}}$ and the corresponding $\{x^i\}$ can play havoc with the resulting notions of differentiability using the path above. That is, continuous functions need not be smooth! This fact is at the heart of the issues of central concern to this book.

Finally, assuming the smoothness issue has been adequately addressed,

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[2] The open ball of radius ϵ centered on x_0^i is defined by $\{x^i | \sum(x^i - x_0^i)^2 < \epsilon^2\}$.

physics needs geometry and more: $\mathcal{G}$, the establishment of a geometry by appending a metric, connection, etc., and $\mathcal{B}$, some sort of bundle structure for defining "internal symmetry" spaces and their connections, etc.[3] These topics will be discussed in the next chapter.

Such questions as these are addressed on the mathematical side by appropriately named *category theory,* which provides an organized way of looking into the study of specific properties such as topology, smoothness, etc., on families of sets. The isomorphisms of a given category correspond to the physical notion of "equivalence" of the corresponding spacetime models. The book by Geroch[Geroch (1985)] provides a useful summary of this and other topics germane to our subject.

1.3 The Basic Questions

The fundamental issues with which we are concerned lie in defining the step $\mathcal{S}$ in 1.1 imposing a smoothness structure on a given topological manifold. However, something like this ambiguity even arises earlier in $\mathcal{T}$ where some point set isomorphisms can drastically alter the topology. As a specific example consider $\mathbb{R}^n_{\mathrm{PS}}$. In the definition of the standard Euclidean topology giving $\mathbb{R}^n_{\mathrm{TOP}}$, the standard metric,

$$d(x, x_0)^2 \equiv \sum_{i=1}^{n} (x^i - x_0^i)^2, \tag{1.2}$$

is used. However, there are clearly infinitely many point set isomorphisms of $\mathbb{R}^n_{\mathrm{PS}}$ with $\mathbb{R}^m_{\mathrm{PS}}$ for any other m. For example, if I is the unit interval, consider $(x, y) \in I^2$. Map $(x, y) \to z \in I^1$ by interlacing the binary digits of x and y. This is clearly a one-to-one map, $I^2 \to I^1$, that is not a homeomorphism. That is, the *topological* notion of dimension is *not* determined by the point-set one apparently inherent in the definition of $\mathbb{R}^n_{\mathrm{PS}}$. Thus, we could impose a topology on $\mathbb{R}^n_{\mathrm{PS}}$ using one such isomorphism of it with $\mathbb{R}^m_{\mathrm{PS}}$ and the Euclidean metric in 1.2 but with n replaced by m. This clearly results in a different topology, $\mathbb{R}^n_{\mathrm{TOP}}$, for the *same* point set $\mathbb{R}^n_{\mathrm{PS}}$. Although there seems little or no obvious motivation for studying such alternative topologies as physical models at the present time, this example does point out ambiguities in definitions that seem so natural as to almost be unique.

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[3] The order in which these two steps are listed in 1.1 is somewhat arbitrary since it could be argued that the imposition of a metric only comes after an establishment of the principal bundle of frames. However, this point is not critical here.

Returning to the issue of defining $\mathcal{S}$ in 1.1, consider the simple example of $n = 1$, so that the topological space is simply the real line with points identified with real numbers and defining the usual Euclidean topology. For clarity, let us use p to denote one such point, so that p is a real number and a point in a topological space. Now, suppose we define a smoothness, $\mathcal{S}$, by defining a global coordinate patch, with coordinate x numerically equal to the real number p,

$$x(p) = p, \quad p(x) = x. \tag{1.3}$$

Here $p \in \mathbb{R}_{\mathrm{TOP}}$ and $x \in \mathbb{R}$, where the latter is the set of real numbers with standard smoothness. Then a function $f : \mathbb{R}_{\mathrm{TOP}} \to \mathbb{R}$ will be smooth if and only if it is C^∞ in the usual real variable sense when expressed in terms of the coordinate, x. If $f_c(x)$ is the coordinate expression, then

$$f_c(x) \equiv f(p(x)). \tag{1.4}$$

Let $\mathcal{F}$ denote this class of smooth functions. Trivially then, the variable x is itself smooth.

However, suppose we had first performed a homeomorphism of $\mathbb{R}_{\mathrm{TOP}}$ onto itself, replacing p by p^3. The topology of the manifold clearly has been unchanged. Now let us define a new global patch and coordinate, y, in terms of this homeomorphic image,

$$y(p) = p^3, \tag{1.5}$$

defining a new smoothness, $\mathcal{S}'$, on the same topological space. Now the class of smooth functions, $\mathcal{F}'$, is defined to be those functions $f'(p)$ such that $f'_c(y) \equiv f'(y^{1/3})$ is C^∞ in the usual sense. Clearly, $\mathcal{F} \neq \mathcal{F}'$. The identity map is an element of $\mathcal{F}$, but, since $y^{1/3}$ is not differentiable at the origin, it is not an element of $\mathcal{F}'$. Thus, we have a simple example of one point set, with two *different* smoothness structures, $\mathcal{S} \neq \mathcal{S}'$.

From the viewpoint of physics, and many mathematical applications, the difference we have established in this example is not *essential.* In fact, mathematicians have found that the most fruitful object for study in differential topology is the equivalence class of smooth structures under diffeomorphisms, that is, homeomorphisms that are smooth when expressed in terms of local coordinates. This equivalence class is the proper object, "differential manifold," with the mathematical category being normally denoted as DIFF. Later chapters will contain more details on this category. For now, let us look at a simple 1-dimensional example provided by the homeomorphism,

$$h(p) = p^{1/3}, \tag{1.6}$$

of $\mathbb{R}_{\mathrm{TOP}}$ onto itself which is surprisingly a diffeomorphism! Its coordinate expression is simply the identity map,

$$h_c(x) \equiv y(h(p(x))) = x. \tag{1.7}$$

Thus, from the viewpoint of differential topology, these two *different* smoothness structures on $\mathbb{R}_{\mathrm{TOP}}$ are actually *diffeomorphic* and equivalent. Diagrammatically,

$$\begin{tikzcd}
\mathbb{R}^1_{TOP} \arrow[r, "x"] \arrow[d, "h"'] & \mathbb{R}^1 \arrow[d, "\mathbb{I}"'] \\
\mathbb{R}^1_{TOP} \arrow[r, "y"] & \mathbb{R}^1
\end{tikzcd} \tag{1.8}$$

is commutative. The two horizontal maps are coordinate maps, defining $\mathcal{S}$ and $\mathcal{S}'$, respectively, the left downward map, h, in 1.6 is a homeomorphism, and the combined map expressed in the two coordinate systems, $\mathbb{I}$, is the identity diffeomorphism. The existence of such a diagram provides the fundamental definition of the equivalence, mathematical *and* physical, of two different smoothness structures.

In parallel with the mathematics, physics after Einstein is thoroughly imbued with the idea from General Relativity that all coordinations of spacetime should be physically equivalent. From the viewpoint of physics, diffeomorphisms are the mathematical embodiment of the notion of coordinate, reference frame, transformations. In other words, from the viewpoint of physics the two structures are equivalent, $\mathcal{S} \sim \mathcal{S}'$.

This discussion leads naturally to some basic questions:

Question 1 (existence): *Does there exist any smoothness structure on a given topological manifold?*

and, if so,

Question 2 (uniqueness): *Is the smoothness structure on a given topological manifold unique up to diffeomorphism?*

We will expend much of our effort on these questions. As stated, Question 2 is too general for present mathematical tools, but special forms of it for restricted classes of topological manifolds, has turned out to be of central importance in recent differential topology and forms the basis for this book. Our main concern will be

Question 3 (Euclidean): *How many smoothness structures (up to diffeomorphisms) can be put on Euclidean topological spaces,* $\mathbb{R}^n_{\mathrm{TOP}}$*?*

To summarize the results discussed more thoroughly in the rest of this book, the answer is

Answer 3(exotic $\mathbb{R}^4_{\mathrm{DIFF}}$**):** *Up to diffeomorphisms, there is one and only one smoothness on* $\mathbb{R}^n_{\mathrm{DIFF}}$ *for* $n \neq 4$*, and uncountably many for the remarkable case of* $n = 4$.

For $\mathbb{R}^1_{\mathrm{DIFF}}$, these assumptions are fairly easy to establish. A very nice and compact proof is provided in the Appendix of Milnor's book[Milnor (1965b)]. For $n = 2, 3$ the proof is much more technical and difficult. For $n > 4$, developments in cobordism theory settled the question. However, until the discoveries of Donaldson, Freedman, et al., the $n = 4$ case remained an open question. Their surprising result for this spacetime dimensional case is the main motivation of this book.

1.4 Some Basic Topological Exotica

1.4.1 *Whitehead continua*

An important thread in understanding exotic smoothness is associated to the assumptions of how properties are inherited in the process of forming mathematical products of spaces. That is, if

$$X = M \times N, \tag{1.9}$$

how are various mathematical properties of M and N related to their point set product, X? Thus, the product formation in 1.9 may be *topological* or *smooth*, etc. Our chief concerns will be in the smooth category, but it is instructive to look at a more basic class of non-intuitive results in low dimensions as provided by *Whitehead continua*[Whitehead (1935)],[Bing (1959)],[Glimm (1960)],[McMillan (1961)].

Whitehead constructed an open, contractible three-dimensional topological manifold, W, which has the following exotic properties:

- W is not homeomorphic to $\mathbb{R}^3$, but,
- $\mathbb{R}^1 \times W$ is homeomorphic to $\mathbb{R}^4$.

In other words, *it is not correct to assume that when an* $\mathbb{R}^1$ *is factored in*

$\mathbb{R}^4$ *the result will necessarily be* $\mathbb{R}^3$.

This too is a profoundly counter-intuitive result. The construction of Whitehead spaces can be visualized using an infinite sequence of twisting tori inside each other. For a discussion with diagrams see [McMillan (1961)]. The limit of the infinite iteration of this process produces a set whose complement in $\mathbb{R}^3$ is a Whitehead space. What the implications of this construction are for the smooth case are not now fully understood, but seem to be highly intriguing. In fact, these spaces are used in handlebody constructions of exotic manifolds.

1.4.2 *Weierstraß functions*

A naive conjecture from elementary calculus is that every function which is continuous over some interval must be at least piecewise smooth, i.e., its derivative exists except at isolated points. "Physical" intuition might well suggest that this conjecture is valid. However, it is not, as demonstrated by the "Weierstraß" functions, such as

$$W(t) = \sum_0^\infty a^k \cos(b^k t), \tag{1.10}$$

where $|a| < 1$. Clearly, this series is absolutely convergent to a continuous function for all t. However, naive term by term differentiation under the summation results in

$$W'(t) \overset{??}{=} -\sum_0^\infty (ab)^k \sin(b^k t). \tag{1.11}$$

If $|ab|$ is chosen to be greater than one, the convergence of this series is dubious at best. In fact, it can be shown rigorously that the derivative of $W(t)$ does not exist anywhere over certain intervals. For more details on such functions see a standard real analysis book such as [Stromberg (1981)].

1.5 The Physics of Certain Mathematical Structures

Finally, let us again come back to the issue of the importance of choice of mathematical structures for physical theories. Einstein's general relativity is nothing other than a theory of the physical importance of the choice of the mathematical structure *geometry.* Later developments indicate that the same may be true with *topology.* In this book we are suggesting that something similar could conceivably be true for the choice of the even more abstract idea of *smoothness*. Because this last concept is more difficult to "visualize" than geometry or even topology, let us conclude this introduction with a toy model of another mathematical structure, *complex structure*.

Our model begins with two-dimensional vacuum electrostatics. Using familiar vector notation, the electrostatic field is represented by a vector on a two-manifold, $\mathbf{E} \in T(M)$, which satisfies

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = 0. \tag{1.12}$$

For simplicity, let $M = \mathbb{R}^2$, and define the *standard* complex structure by

$$z = x + iy, \tag{1.13}$$

and the complex function

$$\mathcal{E} \equiv E_x - iE_y. \tag{1.14}$$

Then, it is well known that the physical equations, 1.12, are equivalent to the complex analysis statement that $\mathcal{E}$ is a holomorphic function, or

$$\frac{\partial}{\partial \bar{z}} \mathcal{E} = 0. \tag{1.15}$$

Now suppose that we decide to express the physical theory of vacuum electrostatics in terms of the statement 1.15. Recall that we are choosing this path to explore the possibility that some analytic structure might influence physics. So, what happens if we *change* the complex structure defined by 1.13. Is there a "complex relativity" in action here? Is the physics dependent on the choice of complex structure?

Before getting into this, let us recall the notion of relativity in the spacetime geometry of standard general relativity. There the basic field can be taken as the metric, which is generally expressed explicitly in terms of components relative to a particular coordinate patch. The basic principle of general relativity then asserts that the more physical expression, "change of reference frames," associated with a change of coordinates leaves the physics itself invariant. Thus, the physical field is represented by the equivalence class of local metric component representations mod coordinate changes. In other words, the physics is not contained in a particular functional form of local coordinate components, but rather in the equivalence class. The "practical" question of whether or not two coordinate presentations of metrics are equivalent is solved by extracting all possible invariant information from the metric. This is a problem that leads to the definition of curvature and its invariants.

So, returning to our toy complex model, suppose we decide that the corresponding complex relativity principle would be that the physics is defined by fields and equations expressed in the same form in different, but biholomorphic (the complex analogue of diffeomorphic) complex structures. So, for example, suppose we choose our complex structure to be defined by

$$z' = x - iy. \tag{1.16}$$

Then clearly z' is not an analytic function of z, but does the physics change? The answer is no, since the underlying recoordination: $(x, y) \rightarrow (x, -y)$ provides a biholomorphism between the z and the z' complex structures. Or, if $\mathbf{F}(x, y) = (x, -y)$, then the statement that the following diagram is commutative with $\mathbb{I}$ being the identity biholomorphism,

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "z"] \arrow[d, "\mathbf{F}"'] & \mathbb{C} \arrow[d, "\mathbb{I}"'] \\ \mathbb{R}^2 \arrow[r, "z'"] & \mathbb{C} \end{tikzcd} \tag{1.17}$$

The problem of determining the equivalence class by evaluating complex invariants is a deep one which we will not go into here. Suffice it to say that the statement

No bound: *every non-constant holomorphic function is unbounded for standard analyticity on* $\mathbb{C}$,

is a well known fact. If we now take the complex structure form of vacuum electrostatics seriously, we would have

No bound(physics): *Every non-constant plane electrostatic vacuum field is unbounded.*

So, can we explicitly find another complex structure for which "No bound" is false? The answer is yes! Let

$$\mathbf{F} : (x, y) \rightarrow (x'', y''), \tag{1.18}$$

be a diffeomorphism of the entire plane onto the unit open ball at the origin. Then

$$z'' = x'' + iy'', \tag{1.19}$$

defines a different complex structure for which the self-defined holomorphic function z'' itself is non-constant and unbounded. Thus, there is no biholomorphism between z'' and the standard z. In terms of the physical interpretation, electrostatics with z'' is truly different from the standard electrostatics, since for this new structure, the "No bound(physics)" statement is false for this z'' complex structure.

1.6 The Physics of Exotic Smoothness

Can differential topology really have anything to do with physical theories? Clearly the answer to this question must be "Yes" because of the principle of general relativity. In light of this principle, the physical content of theories must be invariant under changes of local coordinate patches, *provided that the new smoothness structure is diffeomorphic to the original one.* This is in fact the prototype of "gauge" theory. However, the discovery of exotic smoothness structures shows that there are *many*, often an infinity, of non-diffeomorphic and thus physically inequivalent smoothness structures on many topological spaces of interest to physics. Because of these discoveries, we must face the fact that there is no *a priori* basis for preferring one such structure to another, or to the "standard" one just as we have no *a priori* reason to prefer flat to curved spacetime models. We note that these exotic structures are by definition all *locally* equivalent, so the local expression of physical laws is unchanged. This leads to the apparently paradoxical fact that the implications of exotic smoothness are global, but not in the topological sense!

Unfortunately, the technical difficulties encountered in applying these new results have resulted in only qualitative results for physical applications so far. In the last two chapters we review some of these results and speculate on new ones.

1.7 In Sum

The general question of *equivalence* for various mathematical structures is a fundamental one in mathematics, and, by extension, to physics. Of course defining what "equivalence" should mean is the indispensable first step. Much of the beauty of mathematics is in the discovery of counter examples to intuitively anticipated equivalences, such as the Weierstraß functions and the Whitehead continua examples discussed above.

In general relativity two structures are equivalent if one is obtained from the other by a coordinate transformation (diffeomorphism)[4]. We have long since passed the point in physics of wondering about the principle of general relativity, of saying that two different coordinate statements of the flat metric are physically equivalent while no coordinate transformation can

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[4] Generalizations of this phenomenon lead to "gauge theory" in contemporary mathematical physics.

take a flat metric into a non-flat one. Our toy plane vacuum electrostatic model above displays a similar situation in which a complex structure has replaced geometry, and biholomorphisms have replaced diffeomorphisms: z is equivalent to z' but not to z''.

The motivating factor for this book is the exploration of possible extension of these ideas to the realm of differential or smooth structures. We thought all such were equivalent on $\mathbb{R}^4$. We now know this is not true. Does this present a situation parallel to the discovery of non-euclidean geometries as models for spacetime?