

Preface

This book is about the physics of spacetime at a deep and fundamental level, encoded in the mathematical assumptions of differentiability. We understand the phrase, “physics of spacetime” in the sense that general relativity has taught us, namely that the geometry, the topology, and now the smoothness of our spacetime mathematical models have physical significance.

Our aim is to introduce some of the exciting developments in the mathematics of differential topology over the last fifteen or twenty years to a wider audience than experts. In particular, we are concerned with the discoveries of “exotic” (sometimes called “fake” or “non-standard”) smoothness (differentiable) structures on \mathbb{R}^4 and other topologically simple spaces. We hope to help physicists gain at least a superficial understanding of these results and their potential impact on physical theories involving spacetime models, i.e., *all* fundamental theories. Diffeomorphisms, the basic morphisms of differential topology, are the mathematical representations of the physical notion of transformations between reference frames. As we have learned from Einstein the investigation of these transformations can lead to deep insights into our physical world, as embodied for example in his General Relativistic theory of spacetime and gravity. What the mathematicians have discovered is that the global properties of diffeomorphisms are not at all trivial, even on topologically trivial spaces, such as \mathbb{R}^4 . Yet in general relativity and other field theories physicists continue to assume that the global covering of such spacetime models with smooth reference frames is trivial. This is strongly reminiscent of the assumption of geometric triviality (flatness) of spacetime physics before Einstein. We hope that by presenting an overview of the mathematical discoveries, we may induce physicists to consider the possible physical significance of this newly discovered wealth

of previously unexamined spacetime structures.

Within each section we attempt to accompany the mathematical presentation with parallel narratives of related physical topics as well as more informal “physical” descriptions if feasible. The abundant cross-fertilization of physics and mathematics in recent differential topology makes this endeavor quite natural.

Chapter 1 is an introduction, discussing the traditional interaction of physics and mathematics and speculations that this interaction extends to these differential topology results. We also review some “exotic” and unexpected or at least counter-intuitive facts in elementary topology and analysis to provide somewhat easier analogs to the more technically challenging “exotic” mathematics coming later. We then survey possible physical consequences of these unexpected structures. Chapter 2 begins with a review of some of the mathematical tools and techniques of algebraic topology. Chapter 3 concentrates on the notion of “smoothness” as defined by the introduction of differential structures on topological spaces. The field of differential topology is built on these constructions. In Chapter 4 we provide our first introductory look at some additional structures: bundles, geometry and gauge theory. These topics are considered in more detail later. Chapter 5 delves more deeply into gauge theory and introduces the concepts of moduli space especially those associated with solutions of the Yang-Mills theory. The important, but rather technical, tools associated with surgery and Morse theory are presented in Chapter 6. The remaining chapters then deal with the surprising and we believe physically intriguing discoveries of “exotic” smoothness in the unexpected context of topologically uncomplicated spaces. The first results (1957) along these lines were obtained by Milnor, his exotically smooth seven spheres. These are relatively easy to construct and analyze, and can be immediately associated with physics as having the underlying topology of the bundle of Yang-Mills $SU(2)$ gauge fields over compactified spacetime, S^4 . These topics occupy us in Chapter 7. For the next 20 years or so, much work was done in the field, but some important outstanding questions remained. By various techniques, it was shown that Euclidean topology, \mathbb{R}^n of dimension n can carry only the standard smoothness structures for all $n \neq 4$. The discovery that the missing dimension, 4, the dimension of classical spacetime, concealed a wealth of surprising results came as the results of the work of Freedman, Donaldson et al, having roots in the physically motivated work on moduli space of Yang-Mills connections over a four-manifold. We review the original techniques for studying exotic \mathbb{R}^4 in Chapter 8 and the more current

ones based on Seiberg-Witten theory in Chapter 9. Chapter 10 discusses possible physical implications and Chapter 11 introduces some ideas still in the speculative stage.

We have been warned by many colleagues of the difficulty inherent in writing this book with its mix of recent deep mathematics probably unfamiliar to many physicists and physical topics similarly unfamiliar to mathematicians. Nevertheless, the long and rich history of the interplay of these two disciplines, and a wide variety of books aimed at non-experts on these subjects, makes us believe that this is not a hopeless task. However, we must remind the reader that since the presentation is aimed at “non-experts,” we will necessarily be somewhat superficial and informal in our treatment of many topics, both from the physics and mathematics sides. Full rigor will sometimes be sacrificed, but not at the price of introducing egregious error we hope.

In all of our work we were patiently helped by many colleagues. Especial thanks should go to our mathematician colleagues who showed admirable patience in helping us through the mathematical intricacies leading to the beautiful exotica. Among these we must single out Duane Randall whose patient, expert, and always friendly help was absolutely essential to the completion of this book. We have also received help from others, including R. Gompf, T. Lawson, T. Mautsch, A. Nestke, D. Randall, and H. Rose. We repeat the standard phrase that this work incorporates the knowledge and insights of many people, but any errors are ours alone. This statement is especially true in a book such as this which includes topics from a very wide area. We also were aided by the financial support of a LASpace grant. The second author gratefully acknowledges the hospitality of the Institute for Advanced Studies, Princeton, a Humboldt Senior Research Prize, the hospitality of Friedrich Hehl and others at the Institut für Theoretische Physik, Köln, and the J. C. Carter Professorship at Loyola. Melissa Minneci and Stella VonMeer generously contributed excellent and much needed proof reading assistance. Most importantly, the second author is deeply indebted to Ron Fintushel, then at Tulane, for introducing him to the exciting possibilities of exotic smoothness during stimulating talks. Finally, the second author must affirm the indispensable help of Bill Wilson, Bob Smith and their students, without whom he could not have contributed to this work.