Preface

It often seems to be assumed that there are two kinds of undergraduate linear algebra courses: a course that concentrates on concrete matrix calculations, and an abstract definition-theorem-proof course. It is our conviction that the course needed by most students is neither of these.

For several decades Texas A&M University has taught a course (labeled "Topics in Applied Mathematics I") that combines linear algebra with those advanced topics in multivariable calculus that do not quite fit into the third semester of a calculus sequence.* The audience consists of science and engineering students who have encountered basic multivariable calculus and differential equations and are entering the upper-division courses in their majors. They find the calculational aspects of linear algebra, such as row reduction, determinants, and eigenvectors, rather easy; some have learned them already in other courses. Most of them are not motivated or well prepared to study abstract mathematics for its own sake, or to absorb complete proofs of every theorem — nor is there any reason why they should be.

Most of our effort, therefore, is devoted to introducing modern concepts of linear algebra, which the students may find difficult at first, and demonstrating the usefulness of those concepts, particularly in applied analysis. It is important for students to solve 3 homogeneous linear equations in 4 unknowns; it is important for them to know what the kernel of a linear operator is; it is important to know that the former is an instance of the latter; but the point is lost unless they also recognize other instances, particularly some involving ordinary and partial differential equations. Our philosophy of applied mathematics is†

> *Any blockhead can cite generalities, but the mastermind discerns the particular cases they represent.*

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* For many years the textbook used in this course was *Multivariable Mathematics*, second edition, by R. E. Williamson and H. F. Trotter (Prentice-Hall, 1979). The content and approach of the course and hence of this book have been influenced in many ways by that book. However, there are many important differences.

† Attributed to George Eliot; introduced into mathematical pedagogy by M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Sec. X.1.

It is *the constant interplay between the abstract and the concrete* that makes mathematics such a fascinating subject and such a powerful one. Most students can begin to appreciate this fully only at the upper undergraduate level; unfortunately, this is the level at which the existing textbook literature is most inadequate, and most damagingly compartmentalized.

The history of college mathematics, especially in the United States, has exhibited swings between unsatisfactory extremes. In Ancient Times (before Sputnik), students of science and engineering in their second year of calculus studied partial derivatives and multiple integrals and techniques for solving ordinary differential equations. Some of them went on to take courses in vector calculus (line integrals, curls, etc.) or in partial differential equations and Fourier analysis. Somewhere along the way they usually learned about matrices and how to manipulate them. These things were taught as isolated techniques, with little hint of any connections between them.

In the early 1960s there was a revolution in mathematics teaching, at least in the most selective colleges. Mathematicians began teaching their subject the way they themselves saw it: first the abstract principles, then the concrete applications. The material of second-year calculus was a favorite for this sort of treatment: Students were given definitions of "vector spaces" and "linear operators". Then they were told that

- Matrices are a way of representing the more ethereal — but more fundamental — objects, linear operators. For many purposes it is better to think in terms of linear operators instead of in terms of matrices.
- A partial derivative is one element of a certain matrix associated with a function. The chain rule for functions of several variables is an instance of matrix multiplication.
- The main techniques used to solve linear differential equations ("principles of superposition") are trivial corollaries of the definition of a linear operator (perhaps too trivial to be mentioned explicitly?).

Etc.

When Ph.D.s from the elite universities began teaching students at state universities in the way they had been taught, the result was usually a disaster. The logical order of presentation of mathematics — from the general to the particular — is not always the *psychological* order — the way that best helps most people to learn. Effective pedagogy starts from the concrete, ascends to the abstract, and then returns to the concrete. In recent years, as a response to this problem and also to new computational technology, there has been a reversion to more calculational, less "theoretical", approaches. In linear algebra, to be sure, this sometimes leads into valuable advanced numerical methods. Our concern is that the central importance of linear concepts in applied analysis not get lost in the shuffle. A course in linear algebra for applications should be the entry point into later courses in partial differential equations, Fourier analysis, quantum mechanics, control theory, differential geometry, and applied functional analysis, as well as numerical analysis.

We assume that our students have already learned, at a fairly concrete, pedestrian level, about partial derivatives, linear algebraic equations, geometrical and physical vectors, and elementary ordinary differential equations. We teach (or review) rather quickly the calculational aspects of matrices, eigenvectors, etc. We build on this understanding to develop the general concepts of vector space and linear operator, emphasizing especially the interpretation of abstract algebraic concepts in terms of solvability of homogeneous and nonhomogeneous linear equations, algebraic and differential. Applications to multivariable calculus (tangent vectors, chain rule, surface integrals, etc.) are covered wherever the development of the linear algebra naturally permits. Proofs are emphasized when they build understanding of the concepts, but downplayed when they have no pedagogical value. We hope that this approach enables our students to appreciate linear algebra as the central language and tool of modern applied mathematics.

In summary, a course based on this book sets for its students these learning objectives:

1. To master the calculational techniques of linear algebra, such as row reduction, solving linear algebraic equations, matrix inversion, determinants, constructing bases, Gram–Schmidt orthogonalization, diagonalizing matrices.

2. To become acquainted with the principal abstract concepts of linear algebra, such as linearity, span, linear independence, subspace, kernel and range, inner product, eigenvector; the student should be able to recognize or construct instances of these things in concrete examples (some involving function spaces, not just $\mathbf{R}^n$), should know why they are important in applications, and should acquire at least some intuition for why the main theorems about them are true.

3. To attain deeper mastery of multivariable calculus and differential equations by applying the concepts of linear algebra, as in differentials of nonlinear functions, tangent lines and planes, volumes, line and surface integrals, the chain rule, implicit and inverse functions, local calculations in curvilinear coordinates, solution of systems of ordinary differential equations with constant coefficients, and, above all, the proper treatment of homogeneous and nonhomogeneous differential equations and boundary conditions by linear superposition. In passing, the students review many topics taught in earlier courses but often not mastered there, such as the geometry of lines and planes, determinants and cross products, the calculus of vector functions, spherical coordinates, and the solution of linear differential equations.

Michael Sinyakov and Sergei Tishchenko supplied draft versions of many of the examples and exercises in the book. Also, I am very grateful to those students and faculty colleagues, especially Philip Yasskin, who have corrected mistakes and obtusenesses in the preliminary versions of the book and have suggested various improvements. Finally, encouragement from Jeffrey Morgan and Albert Boggess played important roles in my commitment to this project and completion of it.