- cumulative distribution:

$$F_X(x) = \Phi\left[b \sinh^{-1}\left(\frac{x - m_1}{d}\right)\right]$$

- inverse cumulative distribution:

$$F_X^{-1}(\alpha) = m_1 + d \sinh\left[\frac{\Phi^{-1}(\alpha)}{b}\right] \tag{2}$$

- density function:

$$f_X(x) = \frac{b}{\sqrt{x^2 + d^2}} \phi[b \sinh^{-1}(x/d)]$$

Let $\mu_2(X) = \mu_2$ and $\gamma_2(X) = \gamma_2$ be, respectively, the variance and kurtosis coefficients of the random variable X. If X is a symmetrical Johnson S_U distribution, then we proved in Castagliola (1998) that parameters b and d are related to μ_2 and γ_2 using the following equations (see the appendix for the proof):

$$b = \sqrt{\frac{2}{\ln(\sqrt{2(\gamma_2 + 2)} - 1)}} \tag{3}$$

$$d = \sqrt{\frac{2\mu_2}{\sqrt{2(\gamma_2 + 2)} - 2}} \tag{4}$$

3. Application to Control Charts

Let $X_1, \ldots, X_n$ be a sample of n independent random variables corresponding to training data taken when the process is considered to be "in control", from which we have to compute control limits. Let $\hat{m}_1$, $\hat{\mu}_2$, $\hat{\gamma}_1$, and $\hat{\gamma}_2$ be, respectively, the (moment) estimators of the mean, variance, skewness, and kurtosis. We will assume now (see the appendix) that a first statistical test leads to the conclusion $\gamma_1 = 0$ (the data distribution seems symmetrical) and a second one leads to the conclusion $\gamma_2 \geq 0$ (the data distribution seems leptokurtic). If these conditions are verified, we suggest to compute control limits as presented below:

- Compute $\hat{b}$ and $\hat{d}$ using Eqs. (3) and (4) in which μ_2 and γ_2 have been replaced by their estimators.
- Transform each new observation X to a quasi-normal $N(0, 1)$ random variable Z using the following equation:

$$Z = b \sinh^{-1}\left(\frac{X - m_1}{d}\right) \tag{5}$$

- Use "classical" control limits (mean, median, standard deviation, range, EWMA, CUSUM, etc.) corresponding to a normal $N(0, 1)$ distribution.