7. Conclusions

In this paper we have developed a methodology for the designing of control charts for data having a symmetrical and leptokurtic distribution. We show that this kind of approach is particularly useful when one wants to implement dispersion control charts such as standard deviation or range control charts. The method proposed in this paper allows the quality practitioner to compute "classical" control charts without the use of complex numerical algorithms. The suggested method is so simple that it can be easily programmed into a spreadsheet such as Excel. We also prove that the distributions of the shifted and transformed data can be computed exactly, but this is not the case for the OC and ARL curves which have been obtained by simulation only.

Appendix

How to Test the Skewness and the Kurtosis

The method which is developed in this paper can only be applied if, in theory, the data have a symmetrical leptokurtic distribution. Therefore, in pratice, we need one test to ensure that the distribution of the data is significantly symmetrical, and another test to ensure that the kurtosis of the distribution is significantly positive. In other words, if $\gamma_1 = \mu_3/\mu_2^{3/2}$ and $\gamma_2 = \mu_4/\mu_2^2 - 3$ are the skewness and kurtosis coefficients of the data, we have to solve the two following hypothesis tests

$$\begin{array}{lcl} H_{S0}: \gamma_1 = 0 & \text{and} & H_{K0}: \gamma_2 = 0 \\ H_{S1}: \gamma_1 \neq 0 & & H_{K1}: \gamma_2 > 0 \end{array}$$

Our method can be applied if hypothesis H_{S0} and H_{K1} are accepted. In order to solve the two hypothesis tests above, we suggest the use of an approach which is usually dedicated to test the normality assumption of a data set:

- Compute an estimate for μ

$$\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

- Compute estimates for μ_k, $k = 2, 3, 4$

$$\hat{\mu_k} = \frac{1}{n}\sum_{j=1}^{n} (x_j - \hat{\mu})^k$$

- Compute estimates for γ_1 and γ_2

$$\hat{\gamma_1} = \frac{\hat{\mu_3}}{\hat{\mu_2}^{3/2}} \qquad \text{and} \qquad \hat{\gamma_2} = \frac{\hat{\mu_4}}{\hat{\mu_2}^2} - 3$$

- Compute the standardized statistics

$$\hat{\delta_1} = \frac{\hat{\gamma_1}}{\sqrt{V(\hat{\gamma_1})}} \qquad \text{and} \qquad \hat{\delta_2} = \frac{\hat{\gamma_2}}{\sqrt{V(\hat{\gamma_2})}}$$

- with

$$V(\hat{\gamma_1}) = \frac{6n(n-1)}{(n-2)(n+1)(n+3)}$$

$$V(\hat{\gamma_2}) = \frac{24n(n-1)^2}{(n-3)(n-2)(n+3)(n+5)}$$

- The hypothesis H_{S0} will be accepted with a confidence level $1-\alpha$ if $-\Phi^{-1}(1-\alpha/2) < \hat{\delta_1} < \Phi^{-1}(1-\alpha/2)$.
- The hypothesis H_{K1} will be accepted with a confidence level $1-\alpha$ if $\hat{\delta_2} > \Phi^{-1}(1-\alpha)$.

The symmetry of the distribution can also be tested using the following non-parametric scheme:

- Enumerate the number $\hat{\theta}$ of data which are less than or equal to $\hat{\mu}$.
- The hypothesis H_{S0} will be accepted with a confidence level $1-\alpha$ if $B^{-1}(\alpha/2, n, 0.5) < \hat{\theta} < B^{-1}(1-\alpha/2, n, 0.5)$, where $B^{-1}(\alpha, n, 0.5)$ is the inverse cumulative function of the binomial distribution.

Proof of Eqs. (3) *and* (4)

The moments of order s of the random variable Y can be easily computed using Eq. (1) assuming $g(Y) = \sinh^{-1}(Y)$, i.e.,

$$m_s(Y) = \mu_s(Y) = \frac{1}{2^s\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-z^2/2)[\exp(z/b) - \exp(-z/b)]^s dz$$

In particular, for $s = 2, 4$, we have:

$$\mu_2(Y) = \frac{1}{2}[\exp(2/b^2) - 1]$$

$$\mu_4(Y) = \frac{1}{8}[\exp(8/b^2) - 4\exp(2/b^2) + 3]$$

We deduce the moments of order $s = 2, 4$ of the random variable $X = dY$:

$$\mu_2(X) \stackrel{\text{def}}{=} \mu_2 = \frac{d^2}{2}[\exp(2/b^2) - 1] \tag{15}$$

$$\mu_4(X) \stackrel{\text{def}}{=} \mu_4 = \frac{d^4}{8}[\exp(8/b^2) - 4\exp(2/b^2) + 3]$$

and then:

$$\gamma_2(X) \stackrel{\text{def}}{=} \gamma_2 = \frac{\mu_4(X)}{[\mu_2(X)]^2} - 3 = \frac{\exp(8/b^2) - 4\exp(2/b^2) + 3}{2[\exp(2/b^2) - 1]^2} - 3 \tag{16}$$

If we define $u = \exp(2/b^2)$, then Eqs. (15) and (16) become:

$$\mu_2 = \frac{d^2(u-1)}{2} \tag{17}$$

$$\gamma_2 = \frac{u^4 - 4u + 3}{2(u-1)^2} - 3 \tag{18}$$

We notice that Eq. (18) can be markedly simplified after some basic calculus:

$$\gamma_2 = \frac{1}{2}(u-1)(u+3) \tag{19}$$

The solution for u in Eq. (19) is $u = \sqrt{2(\gamma_2 + 2)} - 1$, and we then deduce the value of b (Eq. (3)). Finally, using Eq. (17) we have d (Eq. (4)).

References

Bai, D. S. and Choi, I. S. (1995), "$\overline{X}$ and R Charts Control Charts for Skewed Populations", *Journal of Quality Technology* **27**, 120–131.

Balakrishnan, N. and Kocherlakota, S. (1986), "Effects of Non-Normality on $\overline{X}$ Charts: Single Assignable Cause Model", *Sankhyā B* **48**, 439–444.

Burr, I. W. (1967), "The Effect of Non-Normality on Constants for $\overline{X}$ and R Charts", *Industrial Quality Control* **24**, 563–569.

Castagliola, P. (1997), "An Improvement of the Weighted Variance $\overline{X}$ Control Chart", in *8th International Symposium on Applied Stochastic Models and Data Analysis*, 49–54.

Castagliola, P. (1998), "Approximation of the Normal Sample Median Distribution Using Symmetrical Johnson S_U Distributions: Application to Quality Control", *Communications in Statistics: Simulation and Computation* **27**(2), 289–301.

Chan, L. K., Hapuarachchi, K. P. and Macpherson, B. D. (1988), "Robustness of $\overline{X}$ and R Charts", *IEEE Transactions on Reliability* **37**, 117–123.

Choobineh, F. and Branting, D. (1986), "A Simple Approximation for Semivariance", *European Journal of Operational Research* **27**, 364–370.

Choobineh, F. and Ballard, J. L. (1987), "Control-Limits of a QC Charts for Skewed Distribution Using Weighted Variance", *IEEE Transactions on Reliability* **36**, 473–477.

Cowden, D. J. (1957), *Statistical Methods in Quality Control*, Prentice-Hall.

Farnum, N. R. (1997), "Using Johnson Curves to Describe Non-Normal Process Data", *Quality Engineering* **9**(2), 329–336.

Ferrell, E. B. (1958), "Control Charts for Log-Normal Universe", *Industrial Quality Control* **15**, 4–6.

Jacobs, D. C. (1990), "Statistical Process Control: Watch out for Non-Normal Distributions", *Chemical Engineering Progress* **86**, 19–27.

Johnson, N. L. (1949), "Systems of Frequency Curves Generated by Methods of Translation", *Biometrika* **36**, 149–176.

Nelson, P. R. (1979), "Control Charts for Weibull Processes with Standards Given", *IEEE Transactions on Reliability* **28**, 283–287.

Pyzdek, T. (1992), "Process Capability Analysis Using Personal Computers", *Quality Engineering* **4**(3), 419–440.

Schilling, E. G. and Nelson, P. R. (1976), "The Effect of Non-Normality on the Control Limits of $\overline{X}$ Charts", *Journal of Quality Technology* **8**, 183–188.

Seppala, T. (1995), "Statistical Process Control Via the Subgroup Bootstrap", *Journal of Quality Technology* **27**, 139–153.

Willemain, T. R. and Runger, G. C. (1996), "Designing Control Charts Using an Empirical Reference Distribution", *Journal of Quality Technology* **28**, 31–38.