

43

Smale and Nonlinear Analysis: A Personal Perspective

ANTHONY J. TROMBA

In the fall of 1966, Smale returned from the International Congress to spend a semester at the Institute for Advanced Study. I had just passed my general exams at Princeton and was looking into a possible thesis topic. As an undergraduate at Cornell, my interest in global nonlinear analysis was ignited by the entertaining and informative lectures of Jim Eells. It was, therefore, natural for me to look for a problem in this area.

Smale's solution to the generalized Poincaré conjecture had already raised him to legendary status among the graduate students who regularly met for their daily tea at Fine Hall. However, the graduate students were also aware of the excitement created by the work of Palais and Smale on an infinite-dimensional version of Morse's critical point theory.

Thus, when Smale lectured at the Institute on his infinite-dimensional version of Sard's theorem and mod(2) degree theory the room was filled to the brim. Although still a mathematical novice, the originality of Smale's viewpoint made a deep impression on me. Smale's attendance at Fine Hall teas then gave me the opportunity to approach him in the crowded and chatty atmosphere of the tea room. In particular I wanted to discuss his cryptic concluding remarks at the lecture that his theory was "related" to Leray-Schauder degree. This remark was eventually to form the core of my and David Elworthy's Ph.D. dissertations. After several conversations, Smale invited me (along with Arthur Greenspoon) to write our theses at Berkeley under his direction.

Upon approaching Salomon Bochner, who was at that time in charge of graduate students, for permission to leave Princeton for Berkeley, he told me "Tromba, we don't care what you do, but if you ever write a thesis just mail it in." With this encouragement, I set out for Berkeley and on my career.

Historically the topology of, and analysis on, infinite-dimensional manifolds began at the Mexico topology conference in 1956 with the presentation by Jim Eells of his fundamental paper showing that suitable function spaces of maps between manifolds themselves form a smooth infinite-dimensional manifold. Jim Eells told the author that although he seemed to be invited everywhere to lecture on his results, no one seemed to pay any additional

attention. However, the papers of Palais and Smale [5, 13] on Morse theory, the paper of Smale [14] on the infinite-dimensional Sard theorem, and the Eells–Sampson breakthrough paper [6] on the existence of harmonic mappings changed things dramatically; suddenly there was enormous interest in the field. The two papers Morse Theory and a Non-Linear Generalization of the Dirichlet Problem, which appeared in the *Annals of Mathematics* in 1964, and An Infinite Dimension Version of Sard's Theorem, which appeared in the *American Journal of Mathematics* in 1965, constitute Smale's main contribution to Nonlinear Analysis.

In the first paper, Smale introduced his (and Palais' [5], who discovered it independently) *condition C*, or what is now commonly called *the Palais–Smale condition*. This work generalized the abstract critical point theory of Marston Morse to infinite-dimensional Hilbert manifolds M equipped with a complete Riemannian metric $\langle, \rangle: TM \times TM \rightarrow R$. Smale considers C^2 -functions $\mathcal{J}: M \rightarrow R$ satisfying the Palais–Smale condition:

Whenever a sequence $\{x_n\} \subset M$ satisfies

- (i) $\mathcal{J}(x_n)$ is bounded,
- (ii) $\|D\mathcal{J}(x_n)\| \rightarrow 0$

[$D\mathcal{J}(x_n)$ the derivative of \mathcal{J} at x_n], then $\{x_n\}$ has a subsequence which converges (naturally to a critical point x_0 of \mathcal{J} ; i.e., $D\mathcal{J}(x_0) = 0$).

Suppose now that $x_0, \mathcal{J}(x_0) = c$, is a nondegenerate critical point [this means that the Hessian, $D^2\mathcal{J}(x_0): T_{x_0}M \times T_{x_0}M \rightarrow R$ induces an isomorphism of $T_{x_0}M$ and its dual space $T_{x_0}M^*$]. Assume further that x_0 is a critical point of finite Morse index θ ; the integer θ is defined to be the dimension of the maximal subspace on which the Hessian is negative definite.¹

Then Smale's main abstract result is that:

$\mathcal{J}^{-1}(-\infty, c + \varepsilon]$ has the homotopy type of $\mathcal{J}^{-1}(-\infty, c - \varepsilon]$ with a cell of dimension θ attached.

Condition *C* then implies the Morse inequalities hold as in finite dimensions, namely,

$$\begin{aligned} C_0 &\geq R_0, \\ C_0 - C_1 &\geq R_1 - R_0, \\ C_2 - C_1 + C_0 &\geq R_2 - R_1 + R_0, \end{aligned}$$

and the Morse equality $\sum (-1)^\theta R_\theta = \sum (-1)^\theta C_\theta$, where R_θ is the Betti number of $\mathcal{J}^{-1}[a, b]$ and C_θ is the number of critical points of index θ in $\mathcal{J}^{-1}[a, b]$ in the level set $\mathcal{J}^{-1}[a, b]$, assuming that a and b are not critical values.

Palais presents a similar theory but shows, in addition, that $\mathcal{J}^{-1}(-\infty, c + \varepsilon)$ is diffeomorphic to $\mathcal{J}^{-1}(-\infty, c - \varepsilon)$ with a cell of dimension θ attached. To

¹ Palais also considers critical points of infinite Morse index and shows that they are topologically irrelevant.

obtain this stronger diffeomorphism result, Palais needs a Hilbert space version of the famous Morse lemma, for which he gives an extremely elegant proof. A simple version of the Morse lemma states that if x_0 is a nondegenerate critical point, one can find a coordinate neighborhood (φ, U) about x_0 so that

$$(\mathcal{J} \circ \varphi^{-1})(x) = Q(x) + \text{constant}$$

for all $x \in \varphi(U)$, where Q is a quadratic form on the Hilbert space which models M . What is interesting and innovative in Smale's approach is that he does not use the Morse lemma, but uses instead only Taylor's theorem to analyze the change of homotopy type near the critical point. This is a fortuitous observation since the Morse lemma and condition C can be incompatible [21]. For example, Struwe's beautiful application of Palais–Smale theory to minimal surfaces [18], and surfaces of constant mean curvature [16, 17], which we later discuss, is exactly such a situation.

Smale concludes his paper by giving a concrete application to a large class of variational problems. For the purpose of exposition, let us consider a special case of Smale's theory, albeit the case that perhaps has the most interest for people in the variational calculus.

Let Ω be a domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. Let $J(Q)$ be the first jet bundle over Ω ; i.e., $J(\Omega) = \Omega \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, where $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ are the linear maps from \mathbb{R}^n to \mathbb{R} . Let $H_0^1(\Omega)$ denote the Sobolev space of functions whose derivatives are square integrable and which assume the Dirichlet boundary value 0 a.e. on $\partial\Omega$. Let $F: J^1(\Omega) \rightarrow \mathbb{R}$ and consider the variational integral $\mathcal{J}: H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(u) = \int_{\Omega} F(x, u(x), Du(x)) dx,$$

where $Du(x)$ denotes the derivative of u at x . Denote the variables of the integrand F by (x, p_0, p_1) or simply (x, p) if we denote the last two variables by one letter. Smale assumes the following conditions:

- (1) $F(x, p) \leq C_1 \|p\|^2 + C_2$,
- (2) $F_{pp}(x, p)(\beta, \beta) \leq C_3 \|\beta\|^2$,
- (3) $C_4 \|p_1\|^2 - C_5 \leq \int_{\Omega} F(x, p) dx$,
- (4) $C_6 \|\beta\|^2 \leq F_{p_1 p_1}(x, p)(\beta, \beta)$,

where subscripts denote partial differentiation. Then if $M = H_0^1(\Omega)$, Smale proves that $\mathcal{J}: M \rightarrow \mathbb{R}$ satisfies condition C . As a consequence, one can conclude that \mathcal{J} has an absolute minimum and, moreover, if the critical points are nondegenerate and finite in number, one can conclude the Morse inequalities

$$\begin{aligned} C_0 &\geq 1, \\ C_1 - C_0 &\geq -1, \\ C_2 - C_1 + C_0 &\geq 1, \end{aligned}$$

and so on.

Note that Smale assumed that the critical points were nondegenerate. At the time, no conditions were known when nondegeneracy held or if nondegeneracy held even in some *generic* situation.

When this paper appeared there was, as I mentioned, an initial burst of excitement. Many mathematicians worldwide looked at it to see if it applied to their problems. After the initial euphoria, people took a more sobering view of these methods. The Palais–Smale condition, in the hands of mathematicians like Paul Rabinowitz and Antonio Ambrosetti, continued to have great success in applications to variational problems with one variable.

However, in the several-variable case, it did not seem to apply to many of the most challenging variational problems of the time, as, for example, the existence of harmonic mappings between Riemannian manifolds, minimal surfaces in space or in Riemannian manifolds, or the existence of H -surfaces, to mention just a few. In fact, Smale told me in 1971 that one of his principle goals in developing the Hilbert manifold Morse theory was to apply it to Plateau's problem, something that Morse himself failed to achieve.

In addition to these criticisms, people were aware that one could obtain the existence of absolute minima under weaker conditions required for a Morse theory, and, in addition, some regularity results and, hence, the existence of classical solutions were known only for minimizers. As a guest of the Steklov Institute in 1969, Anosov and Novikov let me know that they felt there was little future in these methods, which they viewed as just a fancy way of looking at Morse's original ideas. Stefan Hildebrandt, in his interesting modern history of the "Calculus of variations today, reflected in the Oberwolfach Meetings" [10] described the feelings people had in 1968 as follows:

We moreover decided to leave aside the Morse theory developed by Palais and Smale since there seemed to be no applications to interesting geometric problems except for those leading to one-dimensional variational problems.

By 1990, Smale's 60th birthday year, the pendulum had swung back in the other direction. By using condition C for a perturbed variational problem and then going to the limit, Sacks and Uhlenbeck were able to show the existence of minimal spheres in a large class of Riemannian manifolds. Struwe, Reinhold Böhme, and the author were able to complete the Morse theory for minimal surfaces of disc type [2, 15, 22, 23]. Brezis, Coron, and Struwe then used these ideas in solving a conjecture on the existence of large solutions of constant mean curvature [3, 4, 16, 17]. The development of many of these ideas and methods is a long story. But we are getting ahead of ourselves. Let us turn now to Smale's second fundamental paper.

In this paper, Smale was the first to introduce the notion of a nonlinear Fredholm map, i.e., one whose derivative is linear Fredholm, and to define the Fredholm index of such mappings.

Granted, the idea of linear Fredholm maps were very much in the air at this time; Atiyah and Singer had recently proved their famous index theorem

which gave a topological formula for the Fredholm index of a linear elliptic operator defined on sections of vector bundles. Nevertheless, it was Smale's vision to see that much of finite-dimensional differential topology could be carried over to infinite dimensions if one restricted one's attention to Fredholm maps and, furthermore, that these techniques could have significant applications to analysis. The first and most fundamental result in this direction is, of course, Sard's theorem.

Let M and N be second countable C^∞ smooth manifolds with M, N connected. $f: M \rightarrow N$ is said to be *Fredholm* if the sum of the dimensions of the kernel and cokernel of the derivative Df_x is finite for each $x \in M$. This implies that the range of Df_x is closed. In the case f is Fredholm, we define the Fredholm index of f (ind f) by

$$\text{ind } f = \text{index } f = \dim \text{Ker } Df_x - \dim \text{Coker } Df_x.$$

It is an exercise to show that this is independent of $x \in M$. A value $y \in N$ is said to be a regular value if either

(i) $f^{-1}(y) = \emptyset$

or

(ii) if $x \in f^{-1}(y)$, then Df_x is surjective.

The map f is said to be *proper* if the inverse image of compact sets is compact. Then Smale proves the following version of Sard's classic theorem

Theorem (Smale–Sard). *If $f: M \rightarrow N$ is a C^r smooth Fredholm map with $r > \max(\text{ind } f, 0)$, then the set of regular values is a set of second category. If f is proper they are open and dense.*

It was known that the theorem fails if f is not Fredholm or if $r \leq \max(\text{ind } f, 0)$. In addition, counterexamples for real-valued maps on open subsets of Hilbert spaces were also constructed. However, the theorem does hold for real-valued maps f if there exists a "sufficiently smooth Fredholm vector field" X transverse to f ; i.e., $Df_x(X(x)) \geq 0$ where equality holds only if x is simultaneously a critical point of f and a zero of X [26].

Smale's version of Sard's theorem was the beginning of genericity results in nonlinear partial differential equations and the calculus of variations and, in particular, results describing exactly when critical points of functionals \mathcal{J} could be nondegenerate; Smale pointed the way for such results with his own application:

Let $J^2(\Omega)$ be the second jet bundle over a domain $\Omega \subset \mathbb{R}^n$, again with smooth boundary $\partial\Omega$, where

$$J^2(\Omega) = \Omega \times F \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \times \mathcal{L}_s^2(\mathbb{R}^n, \mathbb{R}),$$

where $\mathcal{L}_s^2(\mathbb{R}^n, \mathbb{R})$ are the symmetric bilinear real-valued maps on \mathbb{R}^n . Let $F: J^2(\Omega) \rightarrow \mathbb{R}$ be C^∞ smooth and again denote the variables of F by

(x, p_0, p_1, p_2) . F defines a nonlinear second order operator Φ by

$$\Phi(u) = F(x, u(x), Du(x), D^2u(x)).$$

Let $f: \partial\Omega \rightarrow \mathbb{R}$ and define $C_f^{2,\alpha}(\Omega)$ be the Hölder-space of twice differentiable functions on Ω whose second partials satisfy a Hölder condition with exponent α , $0 < \alpha < 1$, and which agree with f on $\partial\Omega$. Then Φ defines a map from $C_f^{2,\alpha}(\Omega)$ into $C^\alpha(\Omega)$. Smale defines F (or Φ) to be *elliptic* if the derivative $F_{p_2} \in \mathcal{L}_s^2(\mathbb{R}^n, \mathbb{R})$ is a positive or negative definite bilinear form at an points. It then follows from linear elliptic theory that Φ defines a Fredholm map of index zero.

Application of his Sard theorem allows Smale to conclude that for almost all functions $g \in C^\alpha(\Omega)$, the set of solutions to the equation $\Phi(u) = g$ is discrete. Actually, there is a small technical error in Smale’s paper, namely, that $C^{k,\alpha}(\Omega)$ is not a separable Banach space (and so cannot be second countable). Smale should have worked with the space $\Lambda^{k,\alpha}$, the closure of the C^∞ functions in $C^{k,\alpha}$, which is a separable space and for which the Hölder PDE estimates, on which Smale’s theory is based, continue to hold.

Smale concludes his paper by introducing a generalized mod(2) degree for Fredholm maps of non-negative index. If $f: M \rightarrow N$ is a sufficiently smooth and proper (inverse image of compact sets is compact) Fredholm map of index $p > 0$, and N is connected, then the unoriented cobordism class of $f^{-1}(y)$, $[f^{-1}(y)]$, is independent of the choice of regular value y , and $[f^{-1}(y)]$ can thus be interpreted as a generalized mod(2) degree. This idea tied recent developments in topology (the cobordism of René Thom) with developments in non-linear analysis, e.g., degree theory as a tool for existence and uniqueness theorems. In fact, Smale ends his paper, as he did his talk at Princeton, with the remark that the mod(2) degree is related to Leray–Schauder degree. It is safe to say that the full potential of the ideas opened up in this paper are only beginning to be explored.

We would like to conclude this paper with a brief discussion of some results of the author and Michael Struwe which are a culmination of the ideas that Smale initiated in both of his fundamental papers.

Let $D \subset \mathbb{R}^2$ be the unit disc and $\Gamma \subset \mathbb{R}^n$ a smoothly embedded curve, say the image of an H^r smooth embedding $\alpha: \partial D \rightarrow \mathbb{R}^n$. A map $u: D \rightarrow \mathbb{R}^n$, $u \in C^\infty(D^0) \cap C^0(D)$, is said to be a *minimal surface of disc type* spanning Γ if

- (1) $\Delta u = (\Delta u^1, \Delta u^2, \Delta u^n) = 0$,
- (2) $u_x \cdot u_y \equiv 0$,
- (3) $\|u_x\|^2 = \|u_y\|^2$,
- (4) $u: \partial D \rightarrow \Gamma$ is a homeomorphism,

where (x, y) denote the coordinates on D and $\| \cdot \|$ is the \mathbb{R}^n norm.

Conditions (2) and (3) mean that the map is a conformal parameterization of the image surface in \mathbb{R}^n . Since the middle of the nineteenth century, until the work of Jesse Douglas and Tibor Rado in 1931, it was unknown whether or not a map u satisfying (1)–(4) exists for any Γ . Douglas proved existence

by the direct method of the calculus of variations. For this and other work, Douglas was the first (along with Lars Ahlfors) to receive a Fields medal at the International Congress in 1936.

Let \mathcal{N}_α be the manifold of all H^s harmonic maps $u: D \rightarrow \mathbb{R}^n$, $1 < s \ll r$, such that $u|_{\partial D}$ is homotopic to α , and let \mathcal{M}_α denote those $u \in \mathcal{N}_\alpha$ which map ∂D “monotonically” onto $\alpha(\partial D) = \Gamma$ and which map three points on ∂D to three fixed points on Γ ?

Dirichlet's energy $E_\alpha: \mathcal{N}_\alpha \rightarrow \mathbb{R}$ is defined by

$$E_\alpha(u) = \frac{1}{2} \sum_{j=1}^n \int_D \nabla u^j \cdot \nabla u^j dx dy.$$

Roughly speaking, Douglas showed that E_α achieved a minimum on \mathcal{M}_α and that this minimum was a minimal surface.

A natural question is whether a Morse theory holds for $E_\alpha: \mathcal{M}_\alpha \rightarrow \mathbb{R}$ as it does for the geodesics on Riemannian manifolds [5]. Before attempting to answer this question, one would like to know if it is reasonable to assume that the critical points of E_α are nondegenerate in some sense. In this case, to form the manifold \mathcal{N}_α , we must choose $s > 1$. As a consequence, the critical points of E_α cannot possibly be nondegenerate in any classical sense:

First, Dirichlet's energy is invariant under the action of the conformal group \mathcal{G} of the disc, $z \rightarrow c(z - a)/(1 - \bar{a}z)$, $|a| < 1$, $|c| = 1$, a noncompact three-dimensional Lie group. Therefore, the Hessian $D^2E_\alpha(u): T_n\mathcal{N}_\alpha \times T_n\mathcal{N}_\alpha \rightarrow \mathbb{R}$ at a critical point u will always have at least a three-dimensional kernel. Second, and most importantly, if $s > 1$, even after factoring out this kernel, the Hessian cannot induce an isomorphism between a complement to the tangent space of the orbit of \mathcal{G} and its dual space, which is the classical definition of nondegeneracy.

Although the theory of nondegeneracy in this context has been developed [21], for the purpose of this expository paper, we will say that a critical point u of E_α is *classically nondegenerate* if the Hessian $D^2E_\alpha(u)$ has only a three-dimensional kernel. In this case, a Morse lemma holds about such critical points.

In analogy with the theory of geodesics with fixed endpoints, one might conjecture that for almost all α , the critical points of E_α in \mathcal{M}_α are nondegenerate. This, in fact, turns out to be false in dimension $n = 3$. Let us try to understand why this is so, in the context of the ideas originating with Smale.

Minimal surfaces can have branch points, points where $u: D \rightarrow \mathbb{R}^n$ fails to be an immersion. Since $u_x - iu_y: D^0 \rightarrow \mathbb{C}^n$ is holomorphic, each interior branch point $z_0 \in D^0$ has an order λ defined by

$$u_x - iu_y = (z - z_0)^\lambda G(z),$$

where $G(z_0) \neq 0$.

It is less obvious, but nevertheless true, that boundary branch points also have a specific order. By a fundamental result due to Osserman, Gulliver, and Alt, minima do not have interior branch points.

Let Σ be the space of all minimal surfaces in $\bigcup_{\alpha} \mathcal{M}_{\alpha}$ and let Σ_{λ}^{ν} , $\lambda = (\lambda_1, \dots, \lambda_p)$ and $\nu = (\nu_1, \dots, \nu_q)$, be those minimal surfaces with p interior branch points of orders $\lambda_1, \dots, \lambda_p$ and q boundary branch points of orders ν_1, \dots, ν_q . Let $|\lambda| = \sum \lambda_i$ and $|\nu| = \sum \nu_i$.

The following result was proved by the author and R. Böhme [2] and strengthened by Thiel [20]. It is the two-dimensional analogue to Morse's famous theorem that on a Riemannian manifold N for fixed P and for almost all Q , the geodesics joining P and Q are nondegenerate critical points of the energy functional on paths: $\sigma \rightarrow \frac{1}{2} \int \|\sigma'(t)\|^2$. The techniques used to prove this result are, however, drastically different than those used by Morse.

Theorem (Böhme-T). *The strata Σ_{λ}^{ν} are all manifolds; if weighted Sobolev spaces are used then according to Thiel [20], the Σ_{λ}^{ν} are manifolds also. If $\pi: \bigcup_{\alpha} \mathcal{N}_{\alpha} \rightarrow \bigcup_{\alpha} \alpha = \mathcal{A}$ represents the map which sends the surface to its boundary contour and π_{λ}^{ν} the restriction of π to Σ_{λ}^{ν} , then each π_{λ}^{ν} is a nonlinear Fredholm map in the sense of Smale, with index*

$$\text{index } \pi_{\lambda}^{\nu} = 2|\lambda|(2 - n) + |\nu|(2 - n) + 2p + q.$$

In a strong sense $\bigcup \Sigma_{\lambda}^{\nu}$ has the structure of an infinite-dimensional algebraic variety. A generalization of these results to higher genus surfaces has been proved by Tomi and Tromba [19].

For $n \geq 4$, this index is zero only on the stratum Σ_0^0 (i.e., the stratum consisting of no branch points). Let us call a surface $u \in \Sigma_0^0$ *BT nondegenerate* if $D\pi_0^0(u): T_u \Sigma_0^0 \rightarrow T_{\alpha} \mathcal{A}$ is an isomorphism.

As a consequence of the Smale–Sard theorem and the regularity theorem of Hildebrandt [7], almost all $\alpha \in \mathcal{A}$ are regular values of $\pi|_{\bigcup \Sigma_{\lambda}^{\nu}}$, and from the index theorem it follows that an open dense set of α in \mathbb{R}^n have the property that only a finite number of *BT nondegenerate* minimal surfaces span the curve $\alpha(S^1)$, $S^1 = \partial D$. For $n \geq 4$, *BT nondegeneracy* coincides with classical nondegeneracy. Thus, for $n \geq 4$, we know there are only a finite number of minimal surfaces of disc type which span $\alpha(S^1)$ for generic α and all these minimal surfaces will be classically nondegenerate. Let α be such a generic curve, and u_1, \dots, u_N be the disc minimal surfaces spanning $\alpha(S^1)$ of Morse indices $\theta_1, \dots, \theta_N$.² Then the author proved the Morse equality

$$\sum_i (-1)^{\theta_i} = 1. \tag{**}$$

The technique of proof of the author involves a generalization of Leray–Schauder degree to manifolds (in fact, to infinite-dimensional varieties) that Smale hinted at, at the end of his Institute lecture and at the end of the Sard paper. An “analog” of formula (**) is also valid for $n = 3$.

Somewhat later, Struwe [18] was able to use the Palais–Smale condition

² These will always be finite.

suitably adapted to the “convex” set \mathcal{N}_α to show that all Morse inequalities hold; namely,

$$\begin{aligned} C_0 &\geq 1, \\ C_1 - C_0 &\geq -1, \\ C_2 - C_1 + C_0 &\geq 1, \end{aligned}$$

and so on, assuming all critical points are classically nondegenerate, which by the index theorem holds only for $n \geq 4$. As already mentioned, Struwe did not (and could not) use the Morse lemma to derive these inequalities, but fortunately his way (as was the way to generic nondegeneracy) had been paved by Smale.

For $n = 3$, the situation changes dramatically. Here the index is 0 on the stratum Σ_0^0 and all strata Σ_λ^0 , where $\lambda = (1, \dots, 1)$. This implies the existence of simply branched minimal surfaces u such that $D\pi_\lambda^0(u): T_u\Sigma_\lambda^0 \rightarrow T_\alpha\mathcal{A}$ is an isomorphism; i.e., BT nondegenerate. Such minimal surfaces will be stable (i.e., follow along smoothly) under the perturbation of the boundary curve α . However, all such u 's will be classically degenerate since the additional dimension of the kernel of the Hessian at u will be $2|\lambda|$.

Thus, surprisingly, in \mathbb{R}^3 the generic contour does not admit only classically nondegenerate solutions which span it. However again, as a consequence of Smale's generalization of the Sard theorem, the generic contour in \mathbb{R}^3 will admit finitely many BT -nondegenerate minimal surfaces.

The question now arises as to how such a simply branched minimal surface in \mathbb{R}^3 is to be “counted” in the Morse theory of Palais and Smale as adapted to Plateau's problem by Struwe.

From general theoretical principles [21], it follows that there is a nice gradient ∇E_α to $E_\alpha: \mathcal{N}_\alpha \rightarrow \mathbb{R}$. The theory of winding numbers of such vector fields was developed in [24]. The winding numbers of a gradient field around a classically nondegenerate minimal surface will always be ± 1 . However, what about the branched minimal surfaces u in \mathbb{R}^3 which are BT nondegenerate? The surprising answer is that it is not ± 1 , but $\pm 2^p$ where p is the number of branch points for u . Thus, remarkably, the winding number reflects the number of singularities in the surface. As a consequence, it follows that for the Morse inequalities, each such u must be counted as 2^p classically nondegenerate surfaces; and in fact, if the boundary contour for such a surface is perturbed into \mathbb{R}^4 , there will be at least 2^p disc minimal surfaces spanning the perturbed contour.

These results complete Morse's original program of applying his theory to Plateau's problem, a natural geometric variational problem in two variables.

These techniques were further adapted to attack an unsolved problem in the theory of surfaces of constant mean curvature, bounding a contour in \mathbb{R}^3 . For example, cut a smooth region Ω of the topological type of the disc from the sphere S_R^2 of radius R and the remaining region, $S_R^2 - \Omega$ is a surface of constant mean curvature $1/R$ spanning the curve $\partial\Omega$.

Hildebrandt [8] was able to prove a general existence theorem, namely, that if H is a real number and $\Gamma \subset B_R[0]$, the ball of radius R about the origin, and if $|H|R < 1$, then there exists a surface of the type of the disc with mean curvature H and boundary Γ . Such solutions arise as a minimum to a variational problem and are called *small* solutions. Examples indicate that these should also be a second solution, a large H -surface spanning Γ . For, instance, it is easy to see that a simple closed contour on a sphere bounds two solutions.

Struwe [16] (as well as Brezis and Coron [3]) suitably adapting the ideas of Palais and Smale was able to prove the existence of the large solution under the same conditions mentioned for the existence of the small solution.

We cannot fail to mention the wonderful results of Sacks and Uhlenbeck [11, 12] who achieved the first major advance on the existence of harmonic mappings since the work of Eells and Sampson mentioned earlier.

We state two results which follow from their techniques, namely:

Theorem 1. *Let M be a compact surface without boundary, N a compact Riemannian manifold, $g: M \rightarrow N$ continuous. Then there exists a harmonic map $u: M \rightarrow N$ such that the induced maps on the fundamental groups agree; i.e.,*

$$u_{\#} = g_{\#}: \pi_1(M) \rightarrow \pi_1(N).$$

If $\pi_2(N) = 0$, then u is homotopic to g .

Theorem 2. *Suppose the universal cover \tilde{N} of the compact Riemannian manifold N is not contractible. Then there exists a nontrivial conformal harmonic map $u: S^2 \rightarrow N$. Moreover, conformal harmonic maps generate $\pi_2(N)$ as a $Z(\pi_1(N))$ module, and each such harmonic map is energy minimizing in its free homotopy class.*

Harmonic maps are critical points of the Dirichlet energy functional

$$E(v) = \frac{1}{2} \int_M |dv|^2.$$

It is known that E does not satisfy condition C on an appropriate space of manifolds of mappings. Sacks and Uhlenbeck get around this by considering a perturbed functional

$$E_{\alpha}(v) = \frac{1}{2} \int_M (1 + |dv|^2)^{\alpha}$$

for $\alpha > 1$. This does satisfy the Palais–Smale condition and, consequently, has a minimum v_{α} . They obtain their results by letting $\alpha \rightarrow 1$ and showing that v_{α} 's have a convergent subsequence.

These results should be viewed as a triumph of the Palais–Smale theory even though it does not apply directly to the problem at hand.

Finally, we mention another beautiful application of the degree theory that

Smale foresaw, again to minimal surfaces. Brian White [27] has recently shown, using such a theory that minimal surfaces of the type of the torus exist in the three sphere with any metric of positive Ricci curvature.

Surely, the ideas stemming from these two major papers have yet to realize their full potential.

References

- [1] Ambrosetti, A. and Rabinowitz, P., Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
- [2] Böhme, R. and Tromba, A.J., The index theorem for classical minimal surfaces, *Ann. Math.* **113** (1981), 447–499.
- [3] Brézis, H. and Coron, J.M., Sur la conjecture de Rellich pour les surfaces à courbure moyenne prescrite, *C.R. Acad. Sci. Paris Sér. I* **295** (1982), 615–618.
- [4] Brézis, H. and Coron, J.M., Multiple solutions of H -systems and Rellich's conjecture, *Commun. Pure Appl. Math.* **37** (1984), 149–187.
- [5] Palais, R., Morse theory on Hilbert manifolds, *Topology* **2** (1963), 299–340.
- [6] Eells, J. and Sampson, J.H., Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.
- [7] Hildebrandt, S., Boundary behavior of minimal surfaces, *Arch. Rat. Mech. Anal.* **35** (1969), 47–82.
- [8] Hildebrandt, S., On the Plateau problem for surfaces of constant mean curvature, *Commun. Pure Appl. Math.* **23** (1970), 97–114.
- [9] Hildebrandt, S., Über Flächen konstanter mittlerer Krümmung, *Math. Z.* **112** (1969), 107–144.
- [10] Hildebrandt, S., Calculus of variations today, reflected in the Oberwolfach meetings. *Perspective in Mathematics, Anniversary of Oberwolfach 1984*, Birkhäuser, Basel.
- [11] Sacks, J. and Uhlenbeck, K., The existence of minimal immersions of 2-spheres, *Ann. Math.* **113**
- [12] Sacks, J. and Uhlenbeck, K., Minimal immersions of closed Riemann surfaces, *Trans. Amer. Math. Soc.* **271**, 639–652.
- [13] Smale, S., Morse theory and a non-linear generalization of the Dirichlet problem, *Ann. Math. (2)* **80** (1964), 382–396.
- [14] Smale, S., An infinite dimensional version of Sard's theorem, *Amer. J. Math.* **87** (1965), 861–866.
- [15] Struwe, M., On a critical point theory for minimal surfaces spanning a wire in \mathbb{R}^n , *J. Reine Angew. Math.* **349** (1984), 1–23.
- [16] Struwe, M., Large H -surfaces via the mountain-pass-lemma, *Math. Ann.* **270** (1985), 441–459.
- [17] Struwe, M., Nonuniqueness in the Plateau problem for surfaces of constant mean curvature, *Arch. Rat. Mech. Anal.* **93** (1986), 135–157.
- [18] Struwe, M., *Plateau's Problem and the Calculus of Variations*, *Mathematical Notes* No. 35, Princeton University Press, Princeton, NJ, 1988.
- [19] Tomi, F. and Tromba, A.J., The index theorem for minimal surfaces of higher genus, preprint.
- [20] Thiel, U., On the stratification of branched minimal surfaces, *Analysis* **5** (1985), 251–274.

492 A.J. Tromba

- [21] Tromba, A.J., A general approach to Morse theory, *J. Diff. Geom.* **32** (1977), 47–85.
- [22] Tromba, A.J., On the number of minimal surfaces spanning a curve, *Mem. AMS* **194** (1977).
- [23] Tromba, A.J., Degree theory on oriented infinite dimensional varieties and the Morse number of minimal surfaces spanning a curve in \mathbb{R}^n . Part I: $n \geq 4$, *Trans.* **290** (1985), 385–413; Part II: $n = 3$, *Amer. Math. Soc. Manuscripta Math.* **48** (1984), 139–161.
- [24] Tromba, A.J., The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray–Schauder degree, *Adv. Math.* **28** (1978), 148–173.
- [25] Tromba, A.J., Intrinsic third derivatives for Plateau’s problem and the Morse inequalities for disc minimal surfaces in \mathbb{R}^3 , SFB 256, preprint 58, Bonn.
- [26] Tromba, A.J., The Morse–Sard–Brown theorem for functionals and the problem of Plateau, *Amer. J. Math.* **99** (1977), 1251–1256.
- [27] White, B., Every three sphere of positive Ricci curvature contains a minimally embedded torus, *Bull. Amer. Math. Soc.* **21** (1) (1989).