Research Themes

Introduction

Many mathematicians have contributed powerful theorems in several fields. Smale is one of the very few whose work has opened up to research vast areas that were formerly inaccessible. From his early papers in differential topology to his current work in theory of computation, he has inspired and led the development of several fields of research: topology of nonlinear function spaces; structure of manifolds; structural stability and chaos in dynamical systems; applications of dynamical systems to mathematical biology, economics, electrical circuits; Hamiltonian mechanics; nonlinear functional analysis; complexity of real-variable computations. This rich and diverse body of work is outlined in the following subsection.

There are deep connections between Smale's work in apparently disparate fields, stemming from his unusual ability to use creatively ideas from one subject in other, seemingly distant areas. Thus, he used the homotopy theory of fibrations to study immersions of manifolds, and also the classification of differentiable structures. In another area, he applied handle body decompositions of manifolds to structural stability of dynamical systems. Smale applied differential geometry and topology to the analysis of electrical circuits, and to several areas of classical mechanics. He showed how qualitative dynamical systems theory provides a natural framework for investigating complex phenomena in biology. A recent example is his application of algebraic topology to complexity of computation. In each case his innovative approach quickly became a standard research method. His ideas have been further developed by his more than 30 doctoral students, many of whom are now leading researchers in the fields he has pioneered.

This conference brought together mathematicians who are currently making important contributions to these fields. It had two purposes: First, to present recent developments in these fields; and second, to explore the connections between them. This was best done by examining the several areas of Smale's research in a single conference which crosses the traditional boundary lines between mathematical subjects. In this way, a stimulating environment encouraged a fruitful exchange of ideas between mathematicians working in topics that are formally separate, but which, as Smale's work demonstrates, have strong intellectual connections.

Through this conference proceedings we hope that important new insights may be achieved into the extraordinary diversity and unity of mathematics.

Topics

Differential Topology

Smale's first work in differential topology, on the classification of immersions of spheres, led to the general classification of immersions of manifolds. But it also presented, for the first time, the use of fibrations of function spaces in what is now called geometric topology. Through fibrations, the powerful tools of algebraic topology were applied in new ways to a host of geometrical problems. This became, in the hands of Smale and many others, a standard approach to many areas: embeddings, diffeomorphisms, differential structures, piecewise linear theory, submersions, and other fields. The classification of differential structures on topological manifolds due to R. Kirby and L. Siebernmann, and many of the profound geometrical theories of M. Gromov, are based on Smale's technique of function space fibrations.

In 1960, Smale startled the mathematical world with his proofs of the Generalized Poincaré Conjecture and what is now called the H-Cobordism Theorem. Up to that time, the topological classification of manifolds was stuck at dimension three. John Milnor's exciting discovery in 1956 of exotic differential structures on the 7-sphere had pointed to the need for a theory of differential structures, but beyond his examples nothing was known about sufficient conditions for diffeomorphism. Smale had the audacity to attack the problem in dimensions five and above. His results opened the floodgates of research in geometric topology. His techniques of handle cancellation and his constructive use of Morse theory proved enormously fruitful in a host of problems and have become standard approaches to the structural analysis of manifolds. Michael Freedman's recent topological classification of 4-dimensional manifolds is a far-reaching generalization of Smale's handle-canceling methods. It is closely related to exciting developments in Yang–Mills theory by Donaldson, Uhlenbeck, Taubes, and others. This work involves other areas of nonlinear functional analysis and mechanics that will be discussed below.

Dynamical Systems

In the early 1960s, Smale embarked on the study of dynamical systems. Like topology, this subject was founded by Poincaré, who called it the qualitative theory of differential equations. Intensively developed by G.D. Birkhoff, by 1960 it seemed played out as a source of new ideas. At this point, Smale introduced a new approach, based on geometrical assumptions about the dynamical process, rather than the standard method of examining specific equations coming from physics and engineering. The key notion was a hyperbolic structure for the nonwandering set, a far-reaching generalization of the standard notion of hyperbolic fixed point. Under this hypothesis, Smale proved that the nonwandering set (of points that are recurrent in a certain sense) breaks up into a finite number of compact invariant sets in a unique way; these he called basic sets. Each basic set was either a single periodic orbit or contained infinitely many periodic orbits that were tangled in a way that today would be called "chaotic." Moreover, he proved the dynamics in a basic set to be structurally stable.

These new ideas led to a host of conjectures, proofs, examples, and counterexamples by Smale, his many students, and collaborators. Above all, they led to new ways of looking at dynamical systems. These led to precise constructions and rigorous proofs for phenomena that, until then, were only vaguely describable, or only known in very special cases.

For example, Smale's famous Horseshoe is an easily described transformation of the two-dimensional sphere that he proved to be both chaotic (in a precise sense) and structurally stable, and completely describable in combinatorial terms. Moreover, this construction and analysis generalized to all manifolds of all dimensions. But it was more than merely an artificial class of examples, for Smale showed that any system satisfying a simple hypothesis going back to Poincaré (existence of a transverse homoclinic orbit) must have a horseshoe system embedded in it. Such a system is, therefore, not only chaotic, but the chaos is stable in the sense that it cannot be eliminated by arbitrarily small perturbations. In this way, many standard models of natural dynamical processes have been proved to be chaotic.

Smale's new dynamical ideas were quickly applied, by himself and many others, to a variety of dynamical systems in many branches of science.

Nonlinear Functional Analysis

Smale has made fundamental contributions to nonlinear analysis. His application (with R.S. Palais) of Morse's critical point theory to infinite-dimensional Hilbert space has been extensively used for nonlinear problems in both ordinary and partial differential equations. The "Palais–Smale" condition, proving the existence of a critical point for many variational problems, has been used to prove the existence of many periodic solutions for nonlinear Hamiltonian systems. Another application has been to prove the existence of minimal spheres and other surfaces in Riemannian manifolds.

Smale also was a pioneer in the development of the theory of manifolds of maps. The well-known notes of his lectures by Abraham and the related work of Eells has undergone active development ever since. For example, manifolds of maps were used by Arnol'd, Ebin, Marsden, and others in their work on the Lagrangian representation of ideal incompressible fluids, in which the basic configuration space is the group of volume-preserving diffeomorphisms, and for which the Poisson reduced equations are the standard Euler equations of fluid mechanics.

In 1965, Smale proved a generalization of the famous Morse–Sard theorem on the existence of regular values to a wide class of nonlinear mappings in infinite-dimensional Banach spaces. This permitted the use of transversality methods, so useful in finite-dimensional dynamics and topology, for many questions in infinite-dimensional dynamics. An important example is A. Tromba's proof that, generically (in a precise sense), a given simple closed curve in space bounds only a finite number of minimal surfaces of the topological type of the disk. A similar result was proved by Foias and Temam for stationary solutions to the Navier–Stokes equations.

Physical and Biological Applications

Smale's first papers in mechanics are the famous ones on "Topology and Mechanics." These papers appeared in 1970 around the beginning of the geometric formulation of mechanics and its applications, when Mackey's book on the foundations of quantum mechanics and Abraham's book on the foundations of mechanics had just come out. Smale's work centered on the use of topological ideas, principally on the use of Morse theory and bifurcation theory to obtain new results in mechanics. Probably the best-known result in this work concerns relative equilibria in the planar n-body problem, which he obtained by exploiting the topological structure of the level sets of conserved quantities and the reduced phase space, so that Morse theory gave interesting results. For example, he showed that a result of Moulton in 1910, that there are $\frac{1}{2}n!$ collinear relative equilibria, is a consequence of critical point theory. Smale went on to determine the global topology and the bifurcation of the level sets of the conserved angular momentum and the energy for the problem. These papers were a great influence: for example, they led to further work of his former student Palmore on relative equilibria in the planar n-body problem and in vortex dynamics, as well as a number of studies by others on the topology of simple mechanical systems such as the rigid body. This work also was the beginning of the rich symplectic theory of reduction of Hamiltonian systems with symmetry. Smale investigated the case of the tangent bundle with a metric invariant under a group action, which was later generalized and exploited by Marsden, Weinstein, Guillemin, Sternberg, and others for a variety of purposes, ranging from fluids and plasmas to representation theory. The international influence of these papers on a worldwide generation of young workers in the now burgeoning area of geometric mechanics was tremendous.

Smale's work on dynamical systems also had a great influence in mechanics. In particular, the Poincaré–Birkhoff–Smale horseshoe construction has led to studies by many authors with great benefit. For example, it was used by Holmes and Marsden to prove that the PDE for a forced beam has chaotic solutions, by Kopell and Howard to find chaotic solutions in reaction diffusion equations, by Kopell, Varaiya, Marsden, and others in circuit theory, by Levi in forced oscillations, and by Wiggins and Leonard to establish connections between dynamical chaos and Lagrangian or particle mixing rates in fluid mechanics. This construction is regarded as a fundamental one in dynamical systems, and it is also one that is finding the most applications.

In 1972, Smale published his paper on the foundations of electric circuit theory. This paper, highly influenced by an interaction between Smale, Desoer, and Oster, examines the dynamical system defined by the equations for an electric circuit, and gives a study of the invariant sets defined by Kirchhoff's laws and the dynamical systems on these sets. Smale was the first to deal with the implications of the topological complexities that this invariant set might have. In particular, he raised the question of how to deal with the hysteresis or jump phenomena due to singularities in the constraint sets of the form $f(x, dx/dt) = 0$, and he discussed various regularizing devices. This had an influence on the electrical engineering community, such as the 1981 paper of Sastry and Desoer, "The Jump Behavior of Circuits and Systems," which provided the answers to some of the questions raised by Smale's work. (This paper actually originated with Sastry's Masters thesis written in the Department of Mathematics at Berkeley.) This work also motivated the studies of Takens on constrained differential equations.

The best known of Smale's several papers in mathematical biology is the first, in which he constructed an explicit nonlinear example to illustrate the idea of Turing that biological cells can interact via diffusion to create new spatial and/or temporal structure. His deep influence on mathematical biology came less from the papers that he wrote in this field than from the impetus that his pure mathematical work gave to the study of qualitative dynamical systems. Because of the difficulty in measuring all of the relevant variables and the need for clarifying simplification, qualitative dynamical systems provides a natural framework for investigating dynamically complex phenomena in biology. These include "dynamical diseases" (Glass and Mackey), oscillatory phenomena such as neural "central pattern generators" (Ermentrout and Kopell), complexity in ecological equations (May) and immune systems (Perelson), and problems involving spontaneous pattern formation (Howard and Kopell, Murray, Oster).

Another important paper constructed a class of systems of classical competing species equations in $\mathbf{R}^n$ with the property that the simplex Δ^{n-1} spanned by the n coordinate unit vectors is invariant and the trajectories of the large system asymptotic to those of *any* dynamical system in Δ^{n-1}; this demonstrated the possible complexity in systems of competing species. Hirsch has shown that arbitrary systems of competing species decompose into pieces which are virtually identical with Smale's construction. Thus, Smale's example, seemingly a very special case, turns out to be the basic building block for the general case.

Economics

Smale's geometrical approach to dynamics proved fruitful not only in the physical and biological sciences, but also in economic theory. In 1973, he began a series of papers investigating the approach to equilibria in various economic models. In place of the then standard linear methods relying heavily on convexity, he used nonlinear differentiable dynamics with an emphasis on generic behavior. In a sense, this represented a return to an older tradition in mathematical economics, one that relied on calculus rather than algebra, but with intuitive arguments replaced by the rigorous and powerful methods of modern topology and dynamics.

Smale showed that under reasonable assumptions the number of equilibria in a large market economy is generically finite, generalizing work of Debreu. He gave a rigorous treatment of Pareto optimality. His interest in economic processes inspired his work on global Newton algorithms (see the section below). This led to an important paper on price-adjustment processes.

His interest in theoretical economics led Smale to work in the theory of games. In an original approach to the "Prisoner's Dilemma," which is closely related to economic competition, he showed that two players employing certain kinds of reasonable strategies will, in the long run, achieve optimal gains.

Smale's work in economics led to this research in the average stopping time for the simplex algorithm in linear programming, discussed below.

Theory of Computation

Smale's work in economic equlibrium theory led him to consider questions about convergence of algorithms. The economists H. Scarf and C. Eaves had turned Sperner's classical existence proof for the Brouwer fixed-point theorem into a practical computational procedure for approximating a fixed point. Their methods were combinatorial; Smale transformed them into the realm of differentiable dynamics. His "global Newton" method was a simple-looking variant of the classical Newton–Raphson algorithm for solving $f(x) = 0$, where f is a nonlinear transformation of n-space. Unlike the classical Newton's method, which guarantees convergence of the algorithm only if it is started near a solution, Smale proved that, under reasonable assumptions, the global Newton algorithm will converge to a solution for almost every starting point which is sufficiently far from the origin. Because it is easy to find such starting points, this led to algorithms that are guaranteed to converge to a solution with probability one. The global Newton algorithm was the basis for an influential paper on the theory of price adjustment by Smale in mathematical economics.

This geometrical approach to computation was developed further in a series of papers on Newton's method for polynomials, several of which were joint work with M. Shub. These broke new ground by applying to numerical analysis ideas from dynamical systems, differential topology and probability, together with mathematical techniques from many fields: algebraic geometry, geometric measure theory, complex function theory, and differential geometry.

Smale then turned to an algorithm that is of great practical importance, the simplex method for linear programming. It was known that this algorithm usually converged quickly, but that there are pathological examples requiring a number of steps that grows exponentially with the number of variables. Smale asked: What is the average number of steps required for m inequalities in n variables if the coefficients are bounded by 1 in absolute value? He translated this into a geometric problem which he then solved. The surprising answer is that the average number of steps is sublinear in n (or m) if m (or n) is kept fixed.

These new problems, methods, and results led to a great variety of papers by Smale and many others, attacking many questions of computational complexity. Smale always emphasized that he looks at algorithms as mathematicians do, in terms of real numbers, and not as computer scientists do, in terms of a finite number of bits of information. Most recently this led Smale, L. Blum, and M. Shub to a new algebraic approach to the general theory of computability and some surprizing connections with Gödel's theorem.