CONVERGENCE OF GENERALIZED SINGULAR INTEGRALS TO THE UNIT, MULTIVARIATE CASE

George A. Anastassiou and Sorin G. Gal
Department of Mathematical Sciences, The University of Memphis,
Memphis, TN 38152, USA
E-mail: anastasg@msci.memphis.edu, ganastss@memphis.edu
Department of Mathematics, University of Oradea, Str. Armatei Romane 5,
3700 Oradea, Romania
E-mail: galso@math.uoradea.ro

In this article we study the degree of L^p-approximation ($1 \leq p \leq +\infty$) to the unit, by multivariate variants of the Jackson-type generalizations of Picard, Gauss-Weierstrass and Poisson-Cauchy singular integrals.
These results are extensions of those in the univariate case proved by the authors in [2].

1 Introduction

Let f be a function defined on $\mathbf{R}^m$ with values in $\mathbf{R}$.

Let $x = (x_1, \ldots, x_m)$, $h = (h_1, \ldots, h_m) \in \mathbf{R}^m$. Let us denote

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih), \quad r \in \mathbf{N}.$$

We define the rth L^p-modulus of smoothness over $\mathbf{R}^m$, $1 \leq p \leq +\infty$, by

$$\omega_r(f;\delta)_p = \sup\{\|\Delta_h^r f(\cdot)\|_{L^p(\mathbf{R}^m)};\ |h| \leq \delta\},$$

where $|h| = (|h_1|, |h_2|, \ldots, |h_m|)$, $\delta = (\delta_1, \ldots, \delta_m)$, $|h| \leq \delta$ means $|h_i| \leq \delta_i$, $i = \overline{1,m}$ and

$$\|f\|_{L^p(\mathbf{R}^m)} = \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(x_1, \ldots, x_m)|^p dx_1 \ldots dx_m\right\}^{1/p}, \quad \text{if } 1 \leq p < +\infty,$$

$$\|f\|_{L^\infty(\mathbf{R}^m)} = \sup\{|f(x_1, \ldots, x_m)|;\ x_i \in \mathbf{R},\ i = \overline{1,m}\}, \text{ if } p = +\infty.$$

When $f \in L_{2\pi}^p(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{R};\ f$ is 2π-periodic in each variable and $\|f\|_{L_{2\pi}^p(\mathbf{R}^m)} < +\infty\}$, the rth L^p-modulus of smoothness is defined as above using the definitions

$$\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} = \left\{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(x_1, \ldots, x_m)|^p dx_1 \ldots dx_m\right\}^{1/p}, \text{ if } 1 \le p < +\infty,$$
$$\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} = \sup\{|f(x_1, \ldots, x_m)|;\ x_i \in [-\pi, \pi],\ i = \overline{1,m}\}, \text{ if } p = +\infty.$$

Next, for $\xi = (\xi_1, \ldots, \xi_m) > 0$ (i.e. $\xi_i > 0$, $i = \overline{1,m}$), we consider the multivariate variants of the Jackson-type generalizations of Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals introduced in [1] by

$$P_{n,\xi}(f;x) = -\frac{1}{\prod_{i=1}^{m}(2\xi_i)} \sum_{k=1}^{n+1}(-1)^k \binom{n+1}{k} \int_{-\infty}^{+\infty} \cdot$$
$$\cdots \int_{-\infty}^{+\infty} f(x_1 + kt_1, \ldots, x_m + kt_m) \prod_{i=1}^{m} e^{-|t_i|/\xi_i} dt_1 \ldots dt_m,$$

$$Q_{n,\xi}(f;x) = -\frac{1}{\prod_{i=1}^{m}\left[\frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]} \sum_{k=1}^{n+1}(-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi}$$
$$\cdots \int_{-\pi}^{\pi} \frac{f(x_1 + kt_1, \ldots, x_m + kt_m)}{\prod_{i=1}^{m}(t_i^2 + \xi_i^2)} dt_1 \ldots dt_m,$$

and

$$W_{n,\xi}(f;x) = = -\frac{1}{\prod_{i=1}^{m}(2C(\xi_i))} \sum_{k=1}^{n+1}(-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi}$$
$$\cdots \int_{-\pi}^{\pi} f(x_1 + kt_1, \ldots, x_m + kt_m) \prod_{i=1}^{m} e^{-t_i^2/\xi_i^2} dt_1 \ldots dt_m,$$

respectively, where $C(\xi_i) = \int_0^{\pi} e^{-t_i^2/\xi_i^2} dt_i$, $i = \overline{1,m}$, $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$.

Let us denote

$$Erf(x_i) = \frac{2}{\sqrt{\pi}} \int_0^{x_i} e^{-t_i^2} dt_i, \quad x_i \in \mathbf{R}.$$

We notice that

$$\frac{1}{2\xi_i}\int_{-\infty}^{+\infty} e^{-|t_i|/\xi_i} dt_i = 1,$$

$$\int_{-\pi}^{\pi} \frac{dt_i}{t_i^2+\xi_i^2} = \frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right), \quad \xi_i \in \mathbf{R},\ \xi_i > 0$$

and that $\frac{2}{\pi}\tan^{-1}\frac{\pi}{\xi_i}$, $Erf\left(\frac{\pi}{\sqrt{\xi_i}}\right)$ tend to 1 as $\xi_i \to 0$.

The purpose of this article is to study the degree of L^p-approximation ($1 \leq p \leq +\infty$) to the unit, by the above singular integrals, extending those results of the univariate case in [2].

2 Main Results

The first main result is

Theorem 2.1. *Let $X = L^1(\mathbf{R}^m)$ (for $P_{n,\xi}$), $X = L^1_{2\pi}(\mathbf{R}^m)$ (for $W_{n,\xi}, Q_{n,\xi}$), $\xi \in \mathbf{R}^m$, $\xi > 0$, $n \in \mathbf{N}$, $f \in X$.*

Then

$$\|f - P_{n,\xi}(f)\|_X \leq \left[\sum_{k=0}^{n+1}\binom{n+1}{k}k!\right]^m \omega_{n+1}(f;\xi)_X, \quad \xi > 0, \tag{1}$$

$$\|f - W_{n,\xi}(f)\|_X \leq \left[\frac{\displaystyle\int_0^{+\infty}(u+1)^{n+1}e^{-u^2}du}{\displaystyle\int_0^{\pi} e^{-u^2}du}\right]^m \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R}^m)},$$

$$0 \leq \xi \leq 1, \tag{2}$$

$$\|f - Q_{n,\xi}(f)\|_X \leq \left[\prod_{i=1}^{m} K(n,\xi_i)\right]\omega_{n+1}(f;\xi)_X, \quad \xi > 0, \tag{3}$$

where

$$K(n,\xi_i) = \left[\int_0^{\pi/\xi_i}\frac{(u+1)^{n+1}}{u^2+1}du\right] / \tan^{-1}\frac{\pi}{\xi_i}, \quad i = \overline{1,m}.$$

Proof. Let $f \in X = L^1(\mathbf{R}^m)$. We have

$$f(x) - P_{n,\xi}(f;x) = \frac{1}{\prod_{i=1}^{m}(2\xi_i)} \cdot \int_{-\infty}^{+\infty}$$

$$\dots \int_{-\infty}^{+\infty} (-1)^{n+1} \Delta_t^{n+1} f(x) \prod_{i=1}^{m} e^{-|t_i|/\xi_i} dt_1 \dots dt_m, \quad (4)$$

for all $x = (x_1, \dots, x_m)$, $t = (t_1, \dots, t_m)$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbf{R}^m$, $\xi > 0$.

This implies (reasoning as in the univariate case in [2, proof of Theorem 2.1])

$$\|f - P_{n,\xi}(f)\|_{L^1(\mathbf{R}^m)} \le \frac{1}{\prod_{i=1}^{m}(2\xi_i)} \int_{-\infty}^{+\infty}$$

$$\dots \int_{-\infty}^{+\infty} \omega_{n+1}(f; \xi(|t|/\xi))_{L^1(\mathbf{R}^m)} \prod_{i=1}^{m} e^{-|t_i|/\xi_i} dt_1 \dots dt_m,$$

where $\omega_{n+1}(f; \xi(|t|/\xi))_{L^1(\mathbf{R}^m)} := \omega_{n+1}(f; \xi_1(|t_1|/\xi_1), \dots, \xi_m(|t_m|/\xi_m))_{L^1(\mathbf{R}^m)}$.

By denoting $\lambda = \sum_{i=1}^{m} |t_i|/\xi_i \in \mathbf{R}$, we get

$$\begin{aligned}
\omega_{n+1}(f; |t|)_{L^1(\mathbf{R}^m)} &\le \omega_{n+1}(f; \lambda\xi)_{L^1(\mathbf{R}^m)} \\
&\le (\lambda + 1)^{n+1} \omega_{n+1}(f; \xi)_{L^1(\mathbf{R}^m)} \\
&= \left(1 + \sum_{i=0}^{m} |t_i|/\xi_i\right)^{n+1} \omega_{n+1}(f; \xi)_{L^1(\mathbf{R}^m)} \\
&\le \left[\prod_{i=1}^{m}(1 + |t_i|/\xi)\right]^{n+1} \omega_{n+1}(f; \xi)_{L^1(\mathbf{R}^m)}.
\end{aligned}$$

It follows that

$$\|f - P_{n,\xi}(f)\|_{L^1(\mathbf{R}^m)} \le \frac{\omega_{n+1}(f; \xi)_{L^1(\mathbf{R}^m)}}{\prod_{i=1}^{m}(2\xi_i)} \int_{-\infty}^{+\infty}$$

$$\dots \int_{-\infty}^{+\infty} \left(1 + \sum_{i=1}^{m} |t_i|/\xi_i\right)^{n+1} \prod_{i=1}^{m} e^{-|t_i|/\xi_i} dt_1 \dots dt_m$$

$$\le \frac{\omega_{n+1}(f; \xi)_{L^1(\mathbf{R}^m)}}{\prod_{i=1}^{m}(2\xi_i)} \int_{-\infty}^{+\infty}$$

$$\cdots\int_{-\infty}^{+\infty}\left[\prod_{i=1}^{m}(1+|t_i|/\xi_i)\right]^{n+1}\prod_{i=1}^{m}e^{-|t_i|/\xi_i}dt_1\ldots dt_m$$

$$= \text{(see [2, proof of Theorem 2.1])}$$

$$= \left[\sum_{k=0}^{n+1}\binom{n+1}{k}k!\right]^{m}\omega_{n+1}(f;\xi)_{L^1(\mathbf{R}^m)},$$

which proves (1).

Then, for $f \in X = L^1_{2\pi}(\mathbf{R}^m)$, we obtain

$$f(x) - W_{n,\xi}(f;x) = \frac{1}{\prod_{i=1}^{m}(2C(\xi_i))}\int_{-\pi}^{\pi}$$

$$\cdots\int_{-\pi}^{\pi}(-1)^{n+1}\Delta_t^{n+1}f(x)\prod_{i=1}^{m}e^{-t_i^2/\xi_i^2}dt_1\ldots dt_m, \quad (5)$$

for all $x = (x_1,\ldots,x_m)$, $t = (t_1,\ldots,t_m)$, $\xi = (\xi_1,\ldots,\xi_m)$, $\xi_i > 0$, $i = \overline{1,m}$.

Reasoning exactly as in the univariate case in [2, proof of Theorem 2.1], (5) implies (for $0 < \xi_i \le 1$, $i = \overline{1,m}$)

$$\|f - W_{n,\xi}(f)\|_X \le \frac{\omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R}^m)}}{\prod_{i=1}^{m}(2C(\xi_i))}\int_0^{\pi}$$

$$\cdots\int_0^{\pi}\left[\prod_{i=1}^{m}(1+t_i/\xi_i)\right]^{n+1}\prod_{i=1}^{m}e^{-t_i^2/\xi_i^2}dt_1\ldots dt_m$$

$$\le \left[\frac{\int_0^{+\infty}(u+1)^{n+1}e^{-u^2}du}{\int_0^{\pi}e^{-u^2}du}\right]^{m}\omega_{n+1}(f;\xi)_X,$$

which proves (2).

Finally, for $x, t, \xi \in \mathbf{R}^m$, $\xi > 0$, $f \in X = L^1_{2\pi}(\mathbf{R}^m)$, we get

$$f(x) - Q_{n,\xi}(f;x) = \frac{1}{\prod_{i=1}^{m}\left(\frac{2}{\xi_i}\tan^{-1}\frac{\pi}{\xi_i}\right)}\int_{-\pi}^{\pi}$$

$$\dots \int_{-\pi}^{\pi} \frac{(-1)^{n+1}\Delta_t^{n+1} f(x)}{\prod_{i=1}^{m}(t_i^2+\xi_i^2)} dt_1 \dots dt_m, \tag{6}$$

which implies (see the univariate case [2, proof of Theorem 2.1])

$$\|f - Q_{n,\xi}(f)\|_X \le \left[\prod_{i=1}^{m} K(n,\xi_i)\right] \omega_{n+1}(f;\xi)_X,$$

where $K(n,\xi_i) = \left[\int_0^{\pi/\xi_i} \frac{(u+1)^{n+1}}{u^2+1} du\right] / \tan^{-1}\frac{\pi}{\xi_i}$, $i = \overline{1,m}$.

The theorem is proved. □

The second main result is

Theorem 2.2. *Let $X = L^p(\mathbf{R}^m)$ (for $P_{n,\xi}$), $X = L^p_{2\pi}(\mathbf{R}^m)$ (for $W_{n,\xi}, Q_{n,\xi}$), $\xi \in \mathbf{R}^m$, $0 < \xi_i \le 1$, $i = \overline{1,m}$, $n \in \mathbf{N}$, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in X$.*

Then

$$\|f - P_{n,\xi}(f)\|_X \le \left(\frac{2}{q}\right)^{m/q} \|g\|^m_{L^p(\mathbf{R}_+)} \omega_{n+1}(f;\xi)_X,$$

where $g(u) = (u+1)^{n+1}e^{-u/2}$, $u \in \mathbf{R}_+$,

$$\|f - W_{n,\xi}(f)\|_X \le \left[\left(\sqrt{\frac{\pi}{2q}}\right)^{1/q} \|h\|_{L^p(\mathbf{R}_+)} / \int_0^{\pi} e^{-u^2} du\right]^m \omega_{n+1}(f;\xi)_X,$$
$$0 < \xi \le 1,$$

where $h(u) = (u+1)^{n+1}e^{-u^2/2}$,

$$\|f - Q_{n,\xi}(f)\|_X \le \left[\prod_{i=1}^{m} K_p(n,\xi_i)\right] \omega_{n+1}(f;\xi)_X, \quad 0 \le \xi \le 1,$$

where $K_p(n,\xi_i) = \left[\int_0^{\pi/\xi_i} [(u+1)^{(n+1)p}/(u^2+1)] du / \tan^{-1}\frac{\pi}{\xi_i}\right]^{1/p}.$

Proof. Let first $f \in X = L^p(\mathbf{R}^m)$, $\frac{1}{p} + \frac{1}{q} = 1$ and denote $C_1(\xi_i) = \frac{1}{(2\xi_i)^p}\left(\frac{4\xi_i}{q}\right)^{p/q}$. By (4) and similarly with [2, proof of Theorem 2.2], we obtain

$$\|f - P_{n,\xi}(f)\|_X^p \le \prod_{i=1}^{m} C_1(\xi_i) \int_{-\infty}^{+\infty}$$

$$\cdots \int_{-\infty}^{+\infty} \left[\omega_{n+1}(f;|t|)_X \prod_{i=1}^{m} e^{-|t_i|/(2\xi_i)}\right]^p dt_1 \ldots dt_m$$

$$\leq \prod_{i=1}^{m} [2C_1(\xi_i)] \omega_{n+1}^p(f;\xi)_X \int_0^{+\infty}$$

$$\cdots \int_0^{+\infty} \left[\prod_{i=1}^{m}(1+t_i/\xi_i)\right]^{(n+1)p} \prod_{i=1}^{m} e^{-t_i p/(2\xi_i)} dt_1 \ldots dt_m$$

$$= \left(\frac{2}{q}\right)^{m/q} \omega_{n+1}^p(f;\xi)_X \left[\int_0^{+\infty} (u+1)^{(n+1)p} e^{-pu/2} du\right]^m,$$

which implies the first estimate in the statement.

Now, let $f \in X = L_{2\pi}^p(\mathbf{R}^m)$, $\frac{1}{p} + \frac{1}{q} = 1$. By (5) and by the proof of Theorem 2.2 in the univariate case in [2], we obtain

$$\|f - W_{n,\xi}(f)\|^p \leq \left(\prod_{i=1}^{m} [2C_2(\xi_i)]\right) \omega_{n+1}^p(f;\xi)_X$$

$$\cdot \int_0^{\pi} \cdots \int_0^{\pi} \left[\prod_{i=1}^{m}(t_i/\xi_i + 1)\right]^{(n+1)p} \prod_{i=1}^{m} e^{-t_i^2 p/(2\xi_i^2)} dt_1 \ldots dt_m,$$

where

$$C_2(\xi_i) = \frac{1}{[2C(\xi_i)]^p} \left(\sqrt{\frac{2\pi}{q}}\xi_i\right)^{p/q} \left(Erf\left(\pi\sqrt{\frac{q}{2}} \cdot \frac{1}{\xi_i}\right)\right)^{p/q}.$$

Reasoning as in the univariate case (see [2, proof of Theorem 2.2]) we get

$$\|f - W_{n,\xi}(f)\|_X \leq \left[\left(\sqrt{\frac{\pi}{2q}}\right)^{1/q} \|h\|_{L^p(\mathbf{R}_+)} / \int_0^{\pi} e^{-u^2} du\right]^m \omega_{n+1}(f;\xi)_X,$$

where $h(u) = (u+1)^{n+1} e^{-u^2/2}$.

Finally, for $f \in X = L_{2\pi}^p(\mathbf{R}^m)$, $\frac{1}{p} + \frac{1}{q} = 1$, by (6) and by the proof of Theorem 2.2 in [2]) we obtain (omitting the details)

$$\|f - Q_{n,\xi}(f)\|_X^p$$

$$\leq \left(\prod_{i=1}^{m} \left[\int_0^{\pi/\xi_i} ([u+1]^{(n+1)p}/(u^2+1)) du / \tan^{-1} \frac{\pi}{\xi}\right]\right) \omega_{n+1}^p(f;\xi)_X,$$

which proves the theorem. □

The last main result deals with the uniform convergence.

Theorem 2.3. *Let* $X = L^{\infty}(\mathbf{R}^m)$ *(for* $P_{n,\xi}$*),* $X = L^{\infty}_{2\pi}(\mathbf{R}^m)$ *(for* $W_{n,\xi}, Q_{n,\xi}$*),* $\xi \in \mathbf{R}^m$, $0 < \xi \leq 1$, $n \in \mathbf{N}$, $f \in X$. *Then*

$$\|f - P_{n,\xi}(f)\|_X \leq \left[\sum_{k=0}^{m} \binom{n+1}{k} k!\right]^m \omega_{n+1}(f;\xi)_X, \quad \xi > 0,$$

$$\|f - W_{n,\xi}(f)\|_X \leq \left[\frac{\int_0^{+\infty} (u+1)^{n+1} e^{-u^2} du}{\int_0^{\pi} e^{-u^2} du}\right]^m \omega_{n+1}(f;\xi)_X, \quad 0 < \xi \leq 1,$$

$$\|f - Q_{n,\xi}(f)\|_X \leq \left[\prod_{i=1}^{m} K(n,\xi_i)\right] \omega_{n+1}(f;\xi)_X, \quad \xi > 0,$$

where $K(n,\xi_i)$ *is given in Theorem 2.1.*

Proof. By (4), (5), (6) and reasoning exactly as in the proof of Theorem 2.1, we easily obtain the statement of the theorem. □

Remark. Theorems 2.1-2.3 show us that while the generalized operators $P_{n,\xi}$ and $W_{n,\xi}$ give very good estimates (such that if $\xi \to 0$, i.e. $\xi_i \to 0$, $i = \overline{1,m}$, then $P_{n,\xi}(f) \to f$, $W_{n,\xi}(f) \to f$), for the generalized operator $Q_{n,\xi}(f)$ in general this does not happen, because when $\xi_i \to 0$, we have $K_p(n,\xi_i) \to +\infty$, for all $1 \leq p < +\infty$.

However, under some smoothness conditions on f, (as for example if $\left|\dfrac{\partial^{|k|} f}{\partial x_1^{k_1} \dots \partial x_m^{k_m}}\right| \leq M$ on $\mathbf{R}^m$, for all $|k| \in \{0,1,\dots,n+1\}$, where $|k| = k_1 + \dots + k_m$, $k_i \in \mathbf{N} \cup \{0\}$, $i = \overline{1,m}$) and reasoning as in the univariate case [2], we easily obtain that $P_{n,\xi}(f) \to f$, as $\xi \to 0$.

References

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