1. Mathematical Preliminaries

For convenience, the mathematical concepts and results needed to study comparative statics are summarized in this chapter. This is not intended as a comprehensive treatment of these topics. It serves only as a concise summary and convenient reference.

1.1 Differentiable Functions

If the function $y = f(x_1, \ldots, x_n)$ has continuous first-order partial derivatives everywhere in its domain, we say f is C^1. If all partial derivatives up to and including order K exist and are themselves continuous functions, we say f is C^K.

Example 1.1

For the case of $n = 1$, consider the function $y = f(x) = x^{8/3}$. This function is a C^2 function since $f'''(x)$ is undefined at $x = 0$.

1.2 The Chain Rule

Suppose that $y = f(x_1, \ldots, x_n)$ is a C^1 function. Furthermore, suppose that the x's are themselves C^1 functions of the variables a and b, i.e.,

$x_1 = x_1(a, b), \ldots, x_n = x_n(a, b)$.

Then $\dfrac{\partial y}{\partial a} = f_1\dfrac{\partial x_1}{\partial a} + \ldots + f_n\dfrac{\partial x_n}{\partial a}$

and $\dfrac{\partial y}{\partial b} = f_1\dfrac{\partial x_1}{\partial b} + \ldots + f_n\dfrac{\partial x_n}{\partial b}$

where $f_i = \partial f/\partial x_i$, $i = 1, \ldots, n$.

Example 1.2

Suppose $y = f(x_1, x_2) = x_1^2 + x_1x_2$ where $x_1 = a^2$ and $x_2 = a + b$. Then using the Chain Rule, we obtain

$$\begin{aligned}\frac{\partial y}{\partial a} &= (2x_1 + x_2)(2a) + x_1(1)\\ &= (2a^2 + a + b)(2a) + a^2\\ &= 4a^3 + 3a^2 + 2ab\end{aligned}$$

and $\dfrac{\partial y}{\partial b} = (2x_1 + x_2)(0) + x_1(1) = a^2$.

Note that if we substitute $x_1 = a^2$ and $x_2 = a + b$ directly into $x_1^2 + x_1x_2$ we obtain $y = a^4 + a^3 + a^2b$.

Computing $\partial y/\partial a$ and $\partial y/\partial b$ directly yields $\dfrac{\partial y}{\partial a} = 4a^3 + 3a^2 + 2ab$ and $\dfrac{\partial y}{\partial b} = a^2$, which are precisely the results obtained using the Chain Rule.

1.3 Determinants

The *determinant* of the 2 × 2 matrix

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

denoted by $|A|$ is, $|A| = a_{11}\, a_{22} - a_{12}\, a_{21}$. The determinant of the 3 × 3 matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

1.4 Cramer's Rule

Consider the system of equations

$$a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 = b_1$$

$$a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 = b_2$$

$$a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 = b_3.$$

When $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$, the unique solution to this three-equation system is:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|},\ x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|},\ x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|}.$$

1.5 Optimization, f: $R \to R$

Consider the following graph of the function $y = f(x)$:

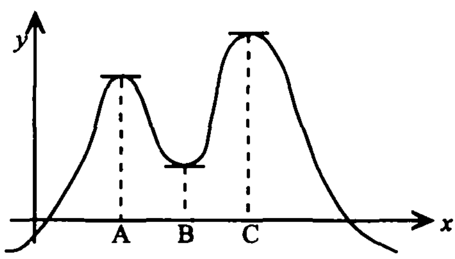

Figure 1.1

Suppose f is C^2. (We shall limit our discussions to consideration of maxima and minima that occur within the interior of the domain of f.) Points A, B, C, are *local extrema*. Points A and C are *local maxima*; point B is a *local minimum*. Note that the slope of the tangent line is zero at each of these points. A *necessary condition* for f to have a local maximum or minimum is $f'(x) = 0$. Any point where $f'(x) = 0$ is called a *critical point*. If, at a critical point $x^*, f''(x^*) < 0$, then x^* is a local maximum. If at a critical point $x^*, f''(x^*) > 0$, then x^* is a local minimum. Note that these are *sufficient conditions*.

1.6 Optimization, $f: R^2 \to R$

Suppose $y = f(x_1, x_2)$ is a C^2 function. A necessary condition for a local maximum or minimum at $\left(x_1^*, x_2^*\right)$ is

$$\frac{\partial f}{\partial x_1} = f_1 = 0, \quad \frac{\partial f}{\partial x_2} = f_2 = 0$$

at this point. This implies that the *tangent plane* at $\left(x_1^*, x_2^*, y^*\right)$ is horizontal. As before, any point where $f_1 = f_2 = 0$ is a critical point. To distinguish local maxima from local minima, form the *Hessian matrix* of second-order partial derivatives

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

If, at a critical point, $f_{11} < 0$ and $|H| > 0$, then this critical point is a local maximum, in which case we say that the Hessian matrix is *negative definite*. If, at a critical point, $f_{11} > 0$ and $|H| > 0$, then this critical point is a local minimum and the Hessian matrix is *positive definite*. These are the higher dimensional analogs of the single variable results discussed in Section 1.5. A local maximum and a local minimum of a function of two variables are depicted in Figures 1.2 and 1.3.

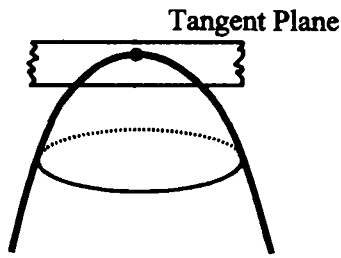

Figure 1.2

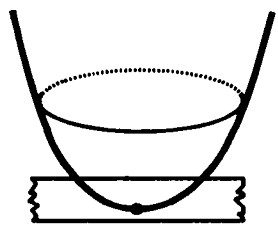

Figure 1.3

Example 1.3.

Consider the function

$$y = f(x_1, x_2) = 10x_1 + 12x_2 - x_1^2 - 6x_2^2.$$

We find the critical point(s) by solving

$$f_1 = 10 - 2x_1 = 0, \quad f_2 = 12 - 12x_2 = 0$$

implying that the unique critical point of f is at $x_1 = 5$, $x_2 = 1$. Noting that

$$H = \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix}$$

with $-2 = f_{11} < 0$ and $|H| = 24 > 0$ establishes the point (5, 1) as a maximum of f. At (5, 1) the graph of this function looks like Figure 1.2.

1.7 Constrained Optimization, Two Variables, One Constraint

Consider the problem of finding the extrema of $f(x_1, x_2)$ subject to $g(x_1, x_2) = 0$ where f and g are C^2. The function f is called the *objective function* and g is called the *constraint*. We are in search of the maxima and minima of f, given that our search must be restricted to the set of points satisfying the equation $g(x_1, x_2) = 0$. To solve the problem, form the *Lagrangian* function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2).$$

The variable λ is called the *Lagrange multiplier*. The first-order necessary conditions for a maximum or minimum are

$$L_1 = f_1 + \lambda g_1 = 0$$

$$L_2 = f_2 + \lambda g_2 = 0 \tag{1}$$

$$L_\lambda = g(x_1, x_2) = 0\,.$$

To distinguish maxima from minima, form the *Bordered Hessian* matrix

$$H = \begin{bmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{bmatrix}.$$

At any point where (1) holds, if $|H| > 0$, then f has a maximum subject to $g(x_1, x_2) = 0$. If, at any point where (1) holds, $|H| < 0$, f has a minimum subject to $g(x_1, x_2) = 0$. Again, these are sufficient conditions.

Example 1.4

Consider

$$\underset{x_1,\, x_2}{\text{Maximize}} \quad x_1^2 x_2$$

subject to $x_1 + 2x_2 - 6 = 0$.

Writing

$$L(x_1, x_2, \lambda) = x_1^2 x_2 + \lambda(x_1 + 2x_2 - 6)$$

the first-order necessary conditions for a maximum are

$$L_1 = 2x_1x_2 + \lambda = 0$$

$$L_2 = x_1^2 + 2\lambda = 0$$

$$L_\lambda = x_1 + 2x_2 - 6 = 0.$$

Solving, one obtains $x_1 = 4$, $x_2 = 1$, $\lambda = -8$. Form the Bordered Hessian

$$H = \begin{bmatrix} 2x_2 & 2x_1 & 1 \\ 2x_1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

which, when evaluated at $x_1 = 4$, $x_2 = 1$ yields

$$H = \begin{bmatrix} 2 & 8 & 1 \\ 8 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

with $|H| > 0$, implying (4, 1) is a constrained maximum of $x_1^2 x_2$. In solving this problem, we found the point on the line $x_1 + 2x_2 = 6$ where the value of $x_1^2 x_2$ is largest.