

2. Phase Space(*)

A geometrical interpretation of Hamilton's equations is obtained by considering a set of values of the dynamical variables p_i and q_i ($i = 1, 2, \dots, n$) as the coordinates of a point P in a $2n$ -dimensional space, called the "phase space."

$$(p_1, q_1, p_2, q_2, \dots, p_n, q_n) \equiv \text{point } P \quad (2.1)$$

in the $2n$ -dimensional phase space

For a system with a single degree of freedom where $n = 1$, phase space can be visualized and represented by a plane (Figure 2.1); although this is not possible for larger n , this representation serves to symbolize the general case. The notation p, q shall be generally understood to stand for the whole set p_i, q_i ($i = 1, 2, \dots, n$).

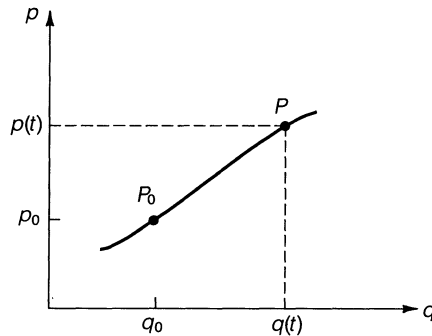


Figure 2.1 Phase space for a system with a single degree of freedom, $n=1$.

Let P_0 represent the point in phase space corresponding to the values p_0, q_0 at the time $t = t_0$. Then, by forward or backward integration of Hamilton's equations, the coordinates p, q of the point P at any other time t are then determined; with t as a continuous variable, the functions $p(t), q(t)$ can be understood to describe a "curve" in phase space, called the *phase orbit*. From the point of view of pure mechanics, the knowledge of these functions or, geometrically, of the motion along the phase orbit, represents, in fact, all that can be said about an individual system. The phase orbit is not necessarily a closed curve, and since a given set of initial values p_0, q_0 completely specifies the subsequent motion of a given mechanical system, phase orbits cannot cross.

It would be necessary, however, to have complete information about the system in the sense that all the values p_0, q_0 are known, or, equivalently, all the values p, q at some other time, in order to determine the motion along a given phase orbit. In practice, this is not normally the case for a macroscopic system where one measures only a few quantities. In thermodynamics, one may not describe a system by more than, say, its pressure and volume or temperature. Suppose the system is composed of $N \cong 10^{23}$ particles. This leaves the very large number $2n = 2 \times 3N$ of data, necessary for the mechanical description, practically unknown and, hence, open to guessing. One is therefore reduced to statistical methods, that is, to a statement of probabilities. In dealing with a mechanical system, it is then necessary, in principle, to consider all possible sets of values p, q that the system may assume at a given time.

Before introducing these statistical methods, however, we prove an important result of pure mechanics, which is known as Liouville's Theorem.

3. Liouville's Theorem(*)

We have seen that the dynamical variables p_i and q_i with $i = 1, 2, \dots, n$ may be regarded as the coordinates in a $2n$ -dimensional space, called the *phase space*. Every set of these variables is thus represented by a point and the quantity

$$d\lambda = \prod_{i=1}^n dp_i dq_i \quad (3.1)$$

represents the *volume element* in the phase space. A mechanical system found at the point $P(p_i, q_i)$ at the time t will follow a definite phase orbit so that at a later time t' it will be found at another point $P'(p'_k, q'_k)$ with the set (p'_k, q'_k) obtained from the initial set (p_i, q_i) by forward integration of Hamilton's equations. Since the new values are determined by the initial values in this fashion, we can write

$$\begin{aligned} p'_k &= p'_k(p_i, q_i) & ; k = 1, 2, \dots, n \\ q'_k &= q'_k(p_i, q_i) \end{aligned} \quad (3.2)$$

These equations can be considered as a transformation of the set of variables (p_i, q_i) to the set (p'_k, q'_k) or, geometrically, as a mapping of the phase space at the time t to that at the time t' . The volume element $d\lambda$ and the corresponding volume element $d\lambda'$, obtained through this transformation (Figure 3.1), are related by the expression

$$d\lambda' = J(t, t')d\lambda \quad (3.3)$$