1.10 List of Symbols

Symbols of frequent use are listed here.

N denote the set of all natural numbers. Namely

$$\mathbf{N} = \{n\ ;\ n = 1, 2, \cdots\}.$$

Z denotes the set of all the integers.
R denotes the set of all real numbers.
C denotes the set of all complex numbers.
H denotes the Hilbert transform in $L^2(S^1)$.
H_η is called the Hilbert transform, too. Here $\eta \in [0, 1)$. $H_0 = H$.

$$H_\eta\left(\sum_{n=1}^{+\infty}(a_n \sin n\sigma + b_n \cos n\sigma)\right) = \sum_{n=1}^{\infty}\frac{1+\eta^{2n}}{1-\eta^{2n}}(-a_n \cos n\sigma + b_n \sin n\sigma).$$

$H^m(S^1)$ denotes the Sobolev spaces, $m = 0, 1, 2, \cdots$.
Function spaces;

$$\begin{aligned} X^m &= \left\{u \in H^m(S^1)\ \middle|\ \int_0^{2\pi} u(\sigma)d\sigma = 0\right\}, \\ Y^m &= \left\{u \in X^m\ \middle|\ \int_0^{2\pi} u(\sigma)\cos k\sigma d\sigma = 0 \quad (\ \forall k \in \mathbf{N}\ )\right\}, \\ Z^m &= \left\{u \in H_0^m(0,\pi)\ \middle|\ \int_0^{\pi} u(\sigma)\cos k\sigma d\sigma = 0 \quad (\ \forall k \in \mathbf{N}\ )\right\}, \end{aligned}$$

where $H_0^m = \{u \in H^m; u(0) = u(\pi) = 0\}$.

Problem

(1) Suppose that the vector field (u, v) satisfies the incompressibility condition $u_x + v_y = 0$. Prove that there exists, in any simply-connected domain, a single-valued function V such that $u = V_y, v = -V_x$.
(2) Suppose that the vector field (u, v) is irrotational: $v_x - u_y = 0$. Prove that there exists, in any simply-connected domain, a single-valued function U such that $u = U_x, v = U_y$.

(3) Let (u, v) be a stationary vector field in the plane satisfying $u_x + v_y = 0$. We define a trajectory of a fluid particle as an integral curve of the following ordinary differential equations:

$$\begin{aligned}\dot{x} &= u(x, y), \\ \dot{y} &= v(x, y).\end{aligned}$$

Prove that any curve defined by $V =$ constant is a trajectory of a fluid particle, where V is a streamfunction.

(4) Derive the Bernoulli equation from the Euler equations under the assumption that the flow is irrotational.

(5) Give an example showing that the Hilbert transform of a bounded function may not be bounded. Hint; consider the function $i \log(1 + z)$ which is analytic in the unit disk.

(6) Give an example showing that the Hilbert transform of a continuous function may not be continuous.

(7) Assume that the free boundary is of C^1 class. Show that there is no stagnation point (= a point where the velocity vector is null) in the velocity field. Hint; Note that the stream function V is zero on the boundary and tends to $-\infty$ as $y \to -\infty$. Use the maximum principle to conclude that ∇V is non-zero anywhere.

(8) Show that the equation (1.37) can be integrated. Hint; define θ and s by

$$\frac{dh}{dx} = \tan\theta, \qquad \frac{dx}{ds} = \cos\theta.$$

Then derive $\theta_{ss} = \alpha \sin\theta$.

(9) Prove (1.24).