1.2 Waves on fluid of finite depth

Thus far, we have considered only flows of infinite depth for the sake of simplicity. However, it is often more natural to consider waves on fluid of finite depth. The work of Levi-Civita [103] was generalized by Struik [173] to the case of finite depth. We now outline his theory.

For the sake of simplicity, we consider only cases where the bottom is a straight horizontal line. In the case where the depth is finite, the conditions (1.5) and (1.7) are retained while (1.8) is replaced by the condition that the bottom bed $\{y = y_0\}$ is a streamline. We write this in the following way:

$$V(x, y_0) \equiv -a \qquad (-L/2 < x < L/2) \tag{1.17}$$

where a is a constant. The constant a and the propagation speed c have the same sign, the proof of which goes as follows. We first remark that U is harmonic and that U satisfies Neumann's boundary condition on the bottom and the free boundary (namely the normal derivative of U is zero, which is equivalent to saying that the normal component of the velocity vanishes on the boundary). We then remark that we may assume $c \neq 0$. To show this, we recall Hopf's maximum principle for harmonic functions (see, e.g., John [77]), which says that the outward normal derivative of a harmonic function at its maxima is positive, unless the function is a constant function. Now, if $c = 0$, then $U \equiv 0$ by Hopf's maximum principle, hence $df/dz \equiv 0$. Consequently, we have

$$gh(x) - \frac{T}{m}\left(\frac{h_x}{\sqrt{1 + h_x^2}}\right)_x = s_0, \tag{1.18}$$

where s_0 is a constant. Putting $h_1(x) = h(x) - s_0/g$, multiplying the equality by h_1, and integrating in $x \in (-L/2, L/2)$, we obtain

$$\int_{-L/2}^{L/2} \left(g h_1^2 + \frac{T}{m} \frac{h_{1,x}^2}{\sqrt{1 + h_{1,x}^2}} \right) dx = 0,$$

from which follows $h_1 \equiv 0$. Here use has been made of the periodicity of h, hence h_1. Also used is the positivity of g and T/m. Now, $h_1 \equiv 0$ implies that the free boundary is flat and the fluid is at rest. Thus we may assume that $c \neq 0$ without losing generality. If that is the case, then U/c takes its maximum on the side boundary $x = L/2$ and its minimum on $-L/2$. On the other hand, we have

$$a = \int_{y_0}^{h(L/2)} \frac{\partial V}{\partial y}(L/2, y)dy = \int_{y_0}^{h(L/2)} \frac{\partial U}{\partial x}(L/2, y)dy.$$

The conclusion now follows from Hopf's maximum principle.

Remark. In deriving "$h \equiv$ constant" from (1.18), we have assumed that g and T/m are of the same sign. This assumption is legitimate when we consider the motion of water in the ocean or a lake. However, if we consider a water layer which is attached to a ceiling, then g is negative while T/m is positive. If that is the case, the triviality of the solution of the equation (1.18) does not hold. See the problem at the end of this chapter. This is also true if the surface tension is negative while g is positive. Those cases where either g or T/m is negative will be dealt with in Chapter 6.

Since both cases are treated similarly, we henceforth assume that both a and c are positive constants. If we define ζ as in (1.11), then ζ runs in $\eta < |\zeta| < 1$, where $\eta = \exp(-2\pi a/cL)$, and ω is an analytic function in the annulus $\{\eta < |\zeta| < 1\}$. The condition (1.17) implies that the velocity is horizontal on $y = y_0$ and that it is expressed as $\theta = 0$ on $\rho = \eta$. Now the problem is:

Find a function $\omega = \omega(\zeta)$ which is continuous on $\{\eta \leq |\zeta| \leq 1\}$, is analytic in $\{\eta < |\zeta| < 1\}$, and satisfies the following (1.19) *and* (1.20).

$$e^{2\tau}\frac{\partial \tau}{\partial \sigma} - pe^{-\tau}\sin\theta + q\frac{\partial}{\partial \sigma}\left(e^{\tau}\frac{\partial \theta}{\partial \sigma}\right) = 0 \qquad \text{on} \quad \rho = 1, \tag{1.19}$$

$$\theta = 0 \qquad \text{on} \quad \rho = \eta. \tag{1.20}$$

In this way the free boundary problem has been transformed to a nonlinear boundary value problem of an analytic function in an annulus. We remark that both Levi-Civita [103] and Struik [173] considered the case where $q = 0$, while Okamoto [132] considered the general case.

As in the case of infinite depth, a further reduction of the equation (1.19) is possible. To this end, the following definition is formulated;

Definition 1.2 For $\eta \in [0,1)$ we define the following operator H_η:

$$H_\eta\left(\sum_{n=1}^{+\infty}(a_n \sin n\sigma + b_n \cos n\sigma)\right) = \sum_{n=1}^{\infty} \frac{1+\eta^{2n}}{1-\eta^{2n}}(-a_n \cos n\sigma + b_n \sin n\sigma). \tag{1.21}$$

Note that H_0 is nothing but H given in Definition 1.1. We call H_η the Hilbert transform, too.

In the case of finite depth, we note that $\tau(1,\sigma) = H_\eta(\theta(1,\sigma)) + \tau_0$, where τ_0 is a constant. If we replace p and q by $pe^{3\tau_0}$ and qe^{τ_0}, respectively, then we have

$$e^{2H_\eta\theta^*}\frac{dH_\eta\theta^*}{d\sigma} - pe^{-H_\eta\theta^*}\sin\theta^* + q\frac{d}{d\sigma}\left(e^{H_\eta\theta^*}\frac{d\theta^*}{d\sigma}\right) = 0 \quad (0 \le \sigma < 2\pi), \tag{1.22}$$

where θ^* denotes $\theta(1,\cdot)$. This form is very convenient in that both cases (finite and infinite depth) are treated in a unified manner. In fact, we obtain (1.16) if we put $\eta = 0$ in (1.22). Therefore the free boundary problem is transformed to the single equation (1.22).

Once $\theta(1,\sigma)$ is known, the free boundary and the bottom are given as in the case of infinite depth. In fact, if a symmetric wave

$$\theta(1,\sigma) = \sum_{n=1}^{\infty} a_n \sin n\sigma$$

is obtained, then

$$\begin{aligned}
\frac{dz}{d\zeta} &= -\frac{L}{2\pi i}\frac{e^{i\omega}}{\zeta}, \\
\theta(\rho,\sigma) &= \sum_{n=1}^{\infty}\frac{\rho^n - \eta^{2n}\rho^{-n}}{1-\eta^{2n}}a_n \sin n\sigma, \\
\tau(\rho,\sigma) &= \sum_{n=1}^{\infty}\frac{-\rho^n - \eta^{2n}\rho^{-n}}{1-\eta^{2n}}a_n \cos n\sigma
\end{aligned}$$

give us the free boundary and the bottom. The formula for the free boundary is actually the same as (1.15)