1.3 Primary bifurcation from the trivial flow

In this section we consider (1.22) and determine the values of (η, p, q) at which solutions other than $\theta^* = 0$ bifurcate. For each positive integer m, we define a Hilbert space X^m by

$$X^m = H^m(S^1)/\mathbf{R} = \left\{ f \in H^m(S^1) \;\middle|\; \int_0^{2\pi} f(\sigma)d\sigma = 0 \right\},$$

where $H^m(S^1)$ is the Sobolev space defined before. Note that $f \in H^m(S^1)$ if and only if f and its derivatives of order $\leq m$ are square integrable. For each $u \in X^2$, we put

$$F(\eta, p, q; u) = e^{2H_\eta u}\frac{dH_\eta u}{d\sigma} - pe^{-H_\eta u}\sin u + q\frac{d}{d\sigma}\left(e^{H_\eta u}\frac{du}{d\sigma}\right). \tag{1.23}$$

Note that $H_\eta u \in X^m$ if $u \in X^m$ and that $H_\eta u \in C^1(S^1)$ if $u \in X^2$, which is a consequence of the Sobolev lemma, see [1], for instance. We do not need here the lemma in full generality. What we need is the following inequality:

$$\max_{0 \leq x \leq 2\pi} |f(x)| \leq \sqrt{\pi}\|f'\|, \tag{1.24}$$

where the prime implies the derivative and $\| \;\; \|$ denotes the L^2-norm, i.e.,

$$\|g\| = \left(\int_0^{2\pi} |g(x)|^2 dx\right)^{1/2}.$$

This fact makes the right hand side of (1.23) square integrable for $u \in X^2$.

Lemma 1.2 *If $u \in X^2$, then $F(\eta, p, q; u) \in X^0$. F is a C^∞-mapping from $[0,1) \times \mathbf{R}^2 \times X^2$ to X^0.*

Proof. By the remark mentioned just above, it is easy to see that F is a C^∞-mapping from $[0,1) \times \mathbf{R}^2 \times X^2$ into $H^0(S^1) = L^2(S^1)$. Hence, we have only to prove that

$$\int_0^{2\pi} F(\eta, p, q; u)d\sigma = 0.$$

Since

$$F(\eta, p, q; u) = \frac{d}{d\sigma}\left(\frac{1}{2}e^{2H_\eta u} + qe^{H_\eta u}\frac{du}{d\sigma}\right) - pe^{-H_\eta u}\sin u,$$

it is sufficient to prove

$$\int_0^{2\pi} e^{-H_\eta u} \sin u d\sigma = 0. \tag{1.25}$$

Note that

$$e^{-H_\eta u} \sin u = \operatorname{Im}\left[e^{i(u+iH_\eta u)} \right].$$

For $X^2 \ni u = \sum_{n=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma)$, we define

$$\omega(\rho,\sigma) = \sum_{n=1}^{\infty} \left(\frac{b_n - ia_n}{1-\eta^{2n}} \zeta^n - \frac{b_n + ia_n}{1-\eta^{2n}} \eta^{2n} \zeta^{-n} \right)$$

which is an analytic function of $\zeta = \rho e^{i\sigma}$ in $\eta < |\zeta| < 1$. It then holds that $\omega(1,\sigma) = u(\sigma) + iH_\eta u(\sigma)$ and $\operatorname{Re}\omega(\eta,\sigma) \equiv 0$. By Cauchy's integral theorem, we have

$$\left(-\int_{\rho=\eta} + \int_{\rho=1} \right) \frac{e^{i\omega}}{\zeta} d\zeta = 0.$$

Taking the real part, we obtain (1.25). □

We thus have a mapping F from $[0,1) \times \mathbf{R}^2 \times X^2$ into X^0 and what we should do is to find zeros of F other than $\{(\eta,p,q;0); 0 \le \eta < 1, 0 \le p < \infty, 0 \le q < \infty\}$. Note that the physical meaning of q requires that $q \in [0,\infty)$. The mapping F, however, has a well-defined mathematical meaning for all $p,q \in \mathbf{R}$. If $p < 0$, we are dealing with a fluid layer attached to a ceiling.

Since F depends on three parameters η, p, and q, the complete description of the set of zeros is not an easy task except for those in a small neighborhood of the trivial solution $u = 0$. As we show below, it is possible to mathematically prove the existence of the bifurcating branch from the trivial solution $u = 0$. On the other hand, it requires formidable calculation to clarify the global structure of the solution set.

In the remaining part of this chapter, we consider the local structure of the bifurcation from trivial solution. By "local", we mean that the solution set in a small neighborhood of the trivial solution is in question. Accordingly, it means waves of small amplitude. On the contrary, "global" means any waves regardless of largeness of amplitude. Solutions near the trivial solution are studied in the remaining part of this chapter and waves of large amplitude are considered, with the aid of numerical experiments, in the subsequent chapters.

The starting point of the bifurcation theory is the study of the linearized operator, which is called the Fréchet derivative in mathematical terminology.

Lemma 1.3 *The Fréchet derivative of* F *at* $u = 0$ *is given by*

$$D_uF(\eta, p, q; 0)w = \frac{d}{d\sigma}H_\eta w - pw + q\frac{d^2w}{d\sigma^2} \qquad (w \in X^2). \tag{1.26}$$

Proof. The Fréchet derivative is characterized by

$$F(\eta, p, q; w) = D_uF(\eta, p, q; 0)w + O(|w|^2) \qquad \text{as } |w| \to 0.$$

It is therefore elementary to prove this lemma from (1.23). □

Corollary 1.1 $D_uF(\eta, p, q; 0) : X^2 \to X^0$ *fails to be isomorphic if and only if* (η, p, q) *satisfies*

$$\frac{1+\eta^{2n}}{1-\eta^{2n}}n - p - n^2q = 0 \tag{1.27}$$

for some positive integer n.

Proof. It is easy to see

$$D_uF(\eta, p, q; 0)(\cos n\sigma) = \left(\frac{1+\eta^{2n}}{1-\eta^{2n}}n - p - n^2q\right)\cos n\sigma.$$

A similar equation holds true for $\sin n\sigma$. The present corollary is proven by expressing u as a Fourier series. □

Remark. The equation (1.27) is identical with what is called the dispersion relation (Lamb[98]).

For fixed n and η, (1.27) defines a line in the (p, q)-plane. We put

$$S_n = \{(\eta, p, q); 0 \le \eta < 1, 0 \le p < \infty, 0 \le q < \infty, (1.27)\}.$$

We notice that two dimensional surface S_n may intersect S_m with $m \ne n$ (see Fig. 1.2).

Definition 1.3 If (η_0, p_0, q_0) belongs to S_n and does not belong to S_m for any $m \ne n$, then we call $(\eta_0, p_0, q_0; 0) \in [0, 1) \times \mathbf{R}^2 \times X^2$ a simple bifurcation

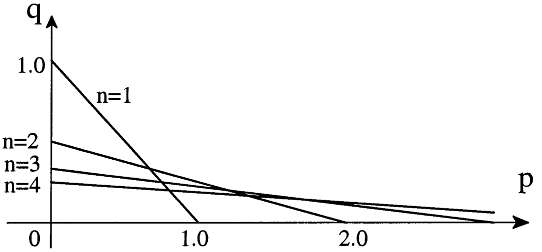

Fig. 1.2 The bifurcation points when $\eta = 0$. The x-axis represents p and the y-axis does q.

point of mode n. If (η_0, p_0, q_0) belongs to S_n and S_m for two different positive integers and does not belong to other S_j with $j \neq m, j \neq n$, then we call (η_0, p_0, q_0) a double bifurcation point of mode (m, n).

The following proposition shows that there is no triple bifurcation points.

Proposition 1.1 *$(\eta, p, q) \in [0,1) \times \mathbf{R}^2$ can satisfy (1.27) for at most two distinct positive integers.*

Since the proof of this proposition is merely a succession of elementary calculations, it is given at the end of this chapter. It is totally unnecessary to read the proof; it is given simply for those readers who want a proof for a proposition or theorem.

Classical results by Levi-Civita[103], Stokes[171], and Struik[173] are included in the following

Theorem 1.1 *If (η_0, p_0, q_0) is a simple bifurcation point, that is,*

$$(\eta_0, p_0, q_0) \in S_n \setminus \bigcup_{m \neq n} S_m, \tag{1.28}$$

then every neighborhood of $(\eta_0, p_0, q_0; 0)$ in $[0,1) \times \mathbf{R}^2 \times X^2$ contains a nontrivial solution.

Proof. We use a closed subspace Y^m of X^m:

$$Y^m = \left\{ f \in X^m \;\middle|\; \int_0^{2\pi} f(\sigma) \cos k\sigma d\sigma = 0 \quad (\text{ for all } k \in \mathbf{N}\ ) \right\}, \tag{1.29}$$

where $\mathbf{N}$ denotes the set of all positive integers. Namely Y^m is the set of all functions in X^m which are odd in $\sigma \in (-\pi, \pi]$. Then it is easy to verify that F is a smooth mapping from $[0,1) \times \mathbf{R}^2 \times Y^2$ into Y^0. Note also that the null space of $D_u F(\eta_0, p_0, q_0; 0)$ is 1-dimensional and spanned by $\sin n\sigma$. We can then use a theorem which ensures the existence of the bifurcation from a simple eigenvalue. The theory of bifurcation of equilibrium is well-established and one can see how it is applied to scientific problems by any standard textbook such as [63, 64, 75, 76]. Here we use the following theorem by Crandall and Rabinowitz [39]:

Theorem 1.2 *Suppose that X and Y are real Banach spaces, V is an open neighborhood of* 0 *in X, and $G : (a,b) \times V \to Y$ is a twice continuously Fréchet differentiable mapping, where (a,b) is an open interval of the real numbers. Suppose that*

(1) $G(\lambda, 0) \equiv 0$ *for* $\lambda \in (a,b)$;
(2) *there exists a* $\lambda_0 \in (a,b)$ *such that* $\dim N(G_u(\lambda_0, 0)) = \operatorname{codim} R(G_u(\lambda_0, 0)) = 1$, (*the subscript implies the Fréchet derivative. N and R denote the null space and the range, respectively.*) ;
(3) $G_{\lambda u}(\lambda_0, 0)x_0 \notin R(G_u(\lambda_0, 0))$, *where* x_0 *spans* $N(G_u(\lambda_0, 0))$.

Let Z be any complement of $\operatorname{Span}\{x_0\}$ *in X, where* $\operatorname{Span}\{x_0\}$ *denotes the one-dimensional space spanned by x_0. Then there exist an open interval I containing* 0 *and continuously differentiable functions* $\lambda : I \to \mathbf{R}$ *and* $\Psi : I \to Z$ *such that* $\lambda(0) = \lambda_0$, $\Psi(0) = 0$, *and* $G(\lambda(s), su_0 + s\Psi(s)) \equiv 0$ *for* $s \in I$. *Furthermore,* $G^{-1}(\{0\})$ *near* $(\lambda_0, 0)$ *consists precisely of the curves* $u = 0$ *and* $(\lambda(s), su_0 + s\Psi(s))$ $(s \in I)$.

Before we apply this theorem, it is worthwhile to understand the meaning of the condition (3) by means of a simple example. Let us consider a smooth mapping of $\mathbf{R}^2$ into $\mathbf{R}$ in the following form: $G(\lambda, x) = \phi(\lambda, x)x$, where x is a real variable. This mapping obviously satisfies the condition (1) in the theorem. In order for this mapping to satisfy the condition (2), we assume that $\phi(0,0) = 0$. Here $\lambda_0 = 0$. Then the condition (3) implies that $\phi_\lambda(0,0) \neq 0$. Now note that the implicit function theorem guarantees the existence of a function $\lambda = \lambda(x)$ defined in a small neighborhood of the origin such that $\phi(\lambda(x), x) = 0$. Therefore we obtain the branch of trivial solutions $(\lambda, 0)$ $(\lambda \in \mathbf{R})$ and the branch of nontrivial solutions $(\lambda(x), x)$ with small x.

In order to use this theorem, we consider any smooth curve $(\eta(\lambda), p(\lambda), q(\lambda))$ $(-1 < \lambda < 1)$ such that $(\eta(0), p(0), q(0)) = (\eta_0, p_0, q_0)$ and the curve is transversal to S_n at (η_0, p_0, q_0). We then define

$$G(\lambda, u) = F(\eta(\lambda), p(\lambda), q(\lambda); u),$$

where u denotes an element of Y^2. Thus, $G : (-1,1) \times Y^2 \to Y^0$ is a smooth mapping with $N(G_u(0,0))$ spanned by $\sin n\sigma$. The range $R(G_u(0,0))$ has a complement spanned by $\sin n\sigma$. By (1.26), we obtain

$$G_{\lambda u}(0,0)(\sin n\sigma) = \frac{d}{d\lambda}\left(\frac{1+\eta(\lambda)^{2n}}{1-\eta(\lambda)^{2n}}n - p(\lambda) - q(\lambda)n^2\right)\sin n\sigma.$$

Thus the condition (3) of Theorem 1.2 is satisfied if

$$\frac{d}{d\lambda}\left(\frac{1+\eta(\lambda)^{2n}}{1-\eta(\lambda)^{2n}}n - p(\lambda) - q(\lambda)n^2\right) \neq 0$$

at $\lambda = 0$. But this is the case precisely when the curve is transversal to S_n. Therefore we are done. □

We show later that double bifurcation points, too, are bifurcation points in the sense that nontrivial solutions exist in every neighborhood of double bifurcation points. This fact was first observed by Wilton [199] and mathematically proved independently by Pierson and Fife [148], Reeder and Shinbrot [151], and Toland and Jones [189, 79]. We give, in Chapter 4, a slightly simpler proof of this fact, which was used in Okamoto [132] by exploiting O(2)-symmetry which will be introduced later.

Before we consider the bifurcation diagram globally, we focus on some special cases where one of the parameters vanishes. In the case that $q = 0$, the solutions are called gravity waves. In the case that $p = 0$, the solutions are called pure capillary waves. In the general case, they are called capillary-gravity waves or gravity-capillary waves. We study pure capillary waves in the next chapter. It is followed by study of gravity waves in Chapter 3 and capillary-gravity waves in Chapter 4.

1.4 Analyticity of the free boundary

The purpose of the present section is to prove that the free boundary is an analytic curve except for a certain limited case. This section is rather technical and those readers who are interested only in phenomenology can