In order to use this theorem, we consider any smooth curve $(\eta(\lambda), p(\lambda), q(\lambda))$ $(-1 < \lambda < 1)$ such that $(\eta(0), p(0), q(0)) = (\eta_0, p_0, q_0)$ and the curve is transversal to S_n at (η_0, p_0, q_0). We then define

$$G(\lambda, u) = F(\eta(\lambda), p(\lambda), q(\lambda); u),$$

where u denotes an element of Y^2. Thus, $G : (-1, 1) \times Y^2 \to Y^0$ is a smooth mapping with $N(G_u(0, 0))$ spanned by $\sin n\sigma$. The range $R(G_u(0, 0))$ has a complement spanned by $\sin n\sigma$. By (1.26), we obtain

$$G_{\lambda u}(0, 0)(\sin n\sigma) = \frac{d}{d\lambda}\left(\frac{1 + \eta(\lambda)^{2n}}{1 - \eta(\lambda)^{2n}} n - p(\lambda) - q(\lambda)n^2\right) \sin n\sigma.$$

Thus the condition (3) of Theorem 1.2 is satisfied if

$$\frac{d}{d\lambda}\left(\frac{1 + \eta(\lambda)^{2n}}{1 - \eta(\lambda)^{2n}} n - p(\lambda) - q(\lambda)n^2\right) \neq 0$$

at $\lambda = 0$. But this is the case precisely when the curve is transversal to S_n. Therefore we are done. □

We show later that double bifurcation points, too, are bifurcation points in the sense that nontrivial solutions exist in every neighborhood of double bifurcation points. This fact was first observed by Wilton [199] and mathematically proved independently by Pierson and Fife [148], Reeder and Shinbrot [151], and Toland and Jones [189, 79]. We give, in Chapter 4, a slightly simpler proof of this fact, which was used in Okamoto [132] by exploiting O(2)-symmetry which will be introduced later.

Before we consider the bifurcation diagram globally, we focus on some special cases where one of the parameters vanishes. In the case that $q = 0$, the solutions are called gravity waves. In the case that $p = 0$, the solutions are called pure capillary waves. In the general case, they are called capillary-gravity waves or gravity-capillary waves. We study pure capillary waves in the next chapter. It is followed by study of gravity waves in Chapter 3 and capillary-gravity waves in Chapter 4.

1.4 Analyticity of the free boundary

The purpose of the present section is to prove that the free boundary is an analytic curve except for a certain limited case. This section is rather technical and those readers who are interested only in phenomenology can skip the present section. The following theorem was obtained by H. Lewy [104]:

Theorem 1.3 *Let the surface tension be neglected and assume that the free boundary is a C^1 (i.e., continuously differentiable) curve. Then it is actually an analytic curve.*

Proof. Let us first notice that the function $U+iV$ is an analytic function defined in Ω_h and ∇V does not vanish on the free boundary. This is proven easily by Hopf's maximum principle (cf. section 1.2). We easily see that $\partial U/\partial s > 0$ everywhere on the free boundary, where s denotes the arclength. Therefore, the analytic function $U+iV$ has an inverse in $-\delta \leq V \leq 0, -cL/2 \leq U \leq cL/2$, where δ is a small positive constant. In this situation, $x+iy$ is an analytic function of $U+iV$ in $-\delta < V < 0, -cL/2 < U < cL/2$, in particular, y and x are harmonic functions of (U,V). If we have shown that y is harmonically extended to $-\delta < V < \eta, -cL/2 < U < cL/2$ with some constant $\eta > 0$, then we know the analyticity of the free boundary curve. Now, this extensibility is guaranteed by the following theorem which was proven by Lewy [104].

Theorem 1.4 *Let $W = W(X,Y)$ be harmonic in $-a < X < a, -b < Y < 0$, where a and b are positive constants. Further, we assume that W is continuously differentiable in $-a \leq X \leq a, -b \leq Y \leq 0$ and that the conjugate harmonic function Z is continuous in $-a \leq X \leq a, -b \leq Y \leq 0$. Suppose that there exists a real analytic function F and that W satisfies the following boundary condition on $Y = 0$:*

$$\frac{\partial W}{\partial Y} = F\left(X, W, Z, \frac{\partial W}{\partial X}\right).$$

Then W and Z are analytically extensible across $Y = 0$.

Note that the Bernoulli condition (1.9) can be written as

$$\left(\frac{\partial y}{\partial U}\right)^2 + \left(\frac{\partial y}{\partial V}\right)^2 = \frac{1}{c_0 - 2gy},$$

where g is the gravity constant and c_0 is a constant. Since we already know that the left hand side is a continuous function, and $\partial y/\partial V$ vanishes nowhere on $V = 0$, we may apply Lewy's theorem above and the proof is complete. □

This theorem asserts that the free boundary is analytic, in particular, infinitely many times differentiable, once its continuous differentiability is known. This does not imply that every possible free boundary is analytic. In fact, we will show in Chapter 3 that the so-called highest wave of G. G. Stokes has a continuous but not differentiable free boundary.

Analyticity of the capillary-gravity waves is similarly proven. In fact the proof is more standard than in the case of the gravity waves, and so it is omitted.

1.5 The case where bottom is not flat

If the flow is infinitely deep, or if the bottom is perfectly flat and horizontal, then we have a trivial solution. If the bottom is not flat, then the very existence of a solution corresponding to the trivial one is a mathematically nontrivial problem. This issue was discussed in Krasovskii [96] and Gerber [59, 60]. The problem has interesting features even in the linearized problem, see, for instance, Article 246 of Lamb [98].

Let us describe the problem in the way employed in Krasovskii [96]. Suppose that the bottom B is represented by

$$(x(s), y(s)) \qquad s \in [0, 1)$$

such that $x(1) = x(0) - L$ and $y(0) = y(1)$. The flow domain $\Omega_{h,B}$ is the simply-connected domain which is bounded by B and $y = h(x)$. The analytic function $x + iy \mapsto U + iV$ is a conformal mapping from $\Omega_{h,B}$ onto

$$\{(U, V) \ ; \ |U| < cL/2, \ -a < V < 0 \}.$$

Define ζ as in (1.11), namely

$$\zeta = \exp\left(-\frac{2\pi i f}{cL}\right).$$

Then

$$\omega = i \log\left(\frac{1}{c}\frac{df}{dz}\right)$$

is an analytic function of ζ defined in $\eta < |\zeta| < 1$. The boundary condition of ω on $|\zeta| = 1$ is just the same as (1.13). The boundary condition on $|\zeta| = \eta$