

This theorem asserts that the free boundary is analytic, in particular, infinitely many times differentiable, once its continuous differentiability is known. This does not imply that every possible free boundary is analytic. In fact, we will show in Chapter 3 that the so-called highest wave of G. G. Stokes has a continuous but not differentiable free boundary.

Analyticity of the capillary-gravity waves is similarly proven. In fact the proof is more standard than in the case of the gravity waves, and so it is omitted.

1.5 The case where bottom is not flat

If the flow is infinitely deep, or if the bottom is perfectly flat and horizontal, then we have a trivial solution. If the bottom is not flat, then the very existence of a solution corresponding to the trivial one is a mathematically non-trivial problem. This issue was discussed in Krasovskii [96] and Gerber [59, 60]. The problem has interesting features even in the linearized problem, see, for instance, Article 246 of Lamb [98].

Let us describe the problem in the way employed in Krasovskii [96]. Suppose that the bottom B is represented by

$$(x(s), y(s)) \quad s \in [0, 1)$$

such that $x(1) = x(0) - L$ and $y(0) = y(1)$. The flow domain $\Omega_{h,B}$ is the simply-connected domain which is bounded by B and $y = h(x)$. The analytic function $x + iy \mapsto U + iV$ is a conformal mapping from $\Omega_{h,B}$ onto

$$\{(U, V) ; |U| < cL/2, -a < V < 0 \}.$$

Define ζ as in (1.11), namely

$$\zeta = \exp \left(-\frac{2\pi i f}{cL} \right).$$

Then

$$\omega = i \log \left(\frac{1}{c} \frac{df}{dz} \right)$$

is an analytic function of ζ defined in $\eta < |\zeta| < 1$. The boundary condition of ω on $|\zeta| = 1$ is just the same as (1.13). The boundary condition on $|\zeta| = \eta$

is generalized as follows. Define $\alpha(s)$ as a continuous function satisfying

$$\tan \alpha(s) = \dot{y}(s)/\dot{x}(s).$$

Without losing generality we assume that $\alpha = 0$ if $\dot{y} = 0$. Then there should exist a monotone increasing continuous function $s = \lambda(\sigma)$ such that

$$\lambda(0) = 0, \quad \lambda(2\pi) = 1, \quad \theta(\eta, \sigma) = \alpha(\lambda(\sigma)).$$

Accordingly,

$$\theta(\eta, \sigma) = \alpha(\lambda(\sigma)) \quad (0 \leq \sigma < 2\pi) \quad (1.30)$$

serves as a boundary condition at $|\zeta| = \eta$.

On the other hand, the function λ is characterized by

$$\frac{d\lambda}{d\sigma} = \frac{Le^{-\tau(\eta, \sigma)}}{2\pi\sqrt{\dot{x}(s)^2 + \dot{y}(s)^2}} \Big|_{s=\lambda(\sigma)}, \quad (1.31)$$

which is derived as follows. As in the way that (1.15) is derived, we obtain the following parametric representation of the bottom:

$$\frac{dx}{d\sigma} = -\frac{L}{2\pi}e^{-\tau(\eta, \sigma)} \cos \theta(\eta, \sigma), \quad \frac{dy}{d\sigma} = -\frac{L}{2\pi}e^{-\tau(\eta, \sigma)} \sin \theta(\eta, \sigma).$$

Since $s = \lambda(\sigma)$, we can derive (1.31).

Therefore, the problem is to find an analytic function ω defined in $\eta < |\zeta| < 1$ satisfying the equation (1.13) on $|\zeta| = 1$ and (1.30) on $|\zeta| = \eta$, and a monotone increasing C^1 function λ satisfying (1.31) in $[0, 1)$.

See also section 3.11.

1.6 Three dimensional waves

Although the present book is devoted to two-dimensional waves, a few comments on three-dimensional waves would be appropriate for better understanding of two-dimensional waves. In three dimensions, we take x -axis as the direction of the propagation of the wave. The y -axis is taken horizontally and perpendicularly to the x -axis. The z -axis is taken vertically upward. The coordinate system is assumed to be moving with the same speed as the wave so that the wave profile is stationary in this coordinate system. The wave profile is assumed to be periodic in both x and y . Let