

is generalized as follows. Define $\alpha(s)$ as a continuous function satisfying

$$\tan \alpha(s) = \dot{y}(s)/\dot{x}(s).$$

Without losing generality we assume that $\alpha = 0$ if $\dot{y} = 0$. Then there should exist a monotone increasing continuous function $s = \lambda(\sigma)$ such that

$$\lambda(0) = 0, \quad \lambda(2\pi) = 1, \quad \theta(\eta, \sigma) = \alpha(\lambda(\sigma)).$$

Accordingly,

$$\theta(\eta, \sigma) = \alpha(\lambda(\sigma)) \quad (0 \leq \sigma < 2\pi) \quad (1.30)$$

serves as a boundary condition at $|\zeta| = \eta$.

On the other hand, the function λ is characterized by

$$\frac{d\lambda}{d\sigma} = \frac{Le^{-\tau(\eta, \sigma)}}{2\pi\sqrt{\dot{x}(s)^2 + \dot{y}(s)^2}} \Big|_{s=\lambda(\sigma)}, \quad (1.31)$$

which is derived as follows. As in the way that (1.15) is derived, we obtain the following parametric representation of the bottom:

$$\frac{dx}{d\sigma} = -\frac{L}{2\pi}e^{-\tau(\eta, \sigma)} \cos \theta(\eta, \sigma), \quad \frac{dy}{d\sigma} = -\frac{L}{2\pi}e^{-\tau(\eta, \sigma)} \sin \theta(\eta, \sigma).$$

Since $s = \lambda(\sigma)$, we can derive (1.31).

Therefore, the problem is to find an analytic function ω defined in $\eta < |\zeta| < 1$ satisfying the equation (1.13) on $|\zeta| = 1$ and (1.30) on $|\zeta| = \eta$, and a monotone increasing C^1 function λ satisfying (1.31) in $[0, 1)$.

See also section 3.11.

1.6 Three dimensional waves

Although the present book is devoted to two-dimensional waves, a few comments on three-dimensional waves would be appropriate for better understanding of two-dimensional waves. In three dimensions, we take x -axis as the direction of the propagation of the wave. The y -axis is taken horizontally and perpendicularly to the x -axis. The z -axis is taken vertically upward. The coordinate system is assumed to be moving with the same speed as the wave so that the wave profile is stationary in this coordinate system. The wave profile is assumed to be periodic in both x and y . Let

L_1 and L_2 denote the wave length in x and y directions, respectively. Let Ω_h denote the three-dimensional domain

$$\{(x, y, z) ; |x| < L_1/2, |y| < L_2/2, -\infty < z < h(x, y) \}.$$

Then we should find a wave profile $z = h(x, y)$ and the potential U such that

$$\Delta U = 0 \quad (\text{in } (x, y, z) \in \Omega_h), \quad (1.32)$$

$$\frac{\partial U}{\partial n} = 0 \quad (\text{on } z = h(x, y)), \quad (1.33)$$

$$\frac{1}{2}|\nabla U|^2 + gz + \frac{T}{m}K = \text{constant} \quad (\text{on } z = h(x, y)), \quad (1.34)$$

$$\lim_{z \rightarrow -\infty} \nabla U = (c, 0, 0), \quad (1.35)$$

where $\partial/\partial n$ denotes the outward normal derivative, $-c$ is the propagation speed, T is the surface tension coefficient, and K is the mean curvature of the free boundary. If we assume that the free boundary is represented as a nonparametric form $z = h(x, y)$, the mean curvature is given by

$$K = -\text{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right).$$

As in the two-dimensional case, z is assumed to be periodic in x and y .

Unlike in the two-dimensional case, we cannot use the streamfunction. Accordingly no transformation of the free boundary problem into a concise form like the Levi-Civita equation is known. Therefore numerical computation accompanies serious difficulties. See, for instance, [118, 121].

1.7 Remarks on equilibrium capillary surface

If we consider the case where the fluid is motionless, then we may set the velocity zero and we obtain an equation which describes equilibrium capillary surfaces. Recall the equation (1.10), which represents a balance among three terms: flow momentum, the gravity, and the capillary force. If the gravity is neglected, namely if $g = 0$, then we obtain the equation for the pure capillary waves, which will be discussed in Chapter 2. If the flow momentum can be neglected (this amount to assuming that g and T are