

L_1 and L_2 denote the wave length in x and y directions, respectively. Let Ω_h denote the three-dimensional domain

$$\{(x, y, z) ; |x| < L_1/2, |y| < L_2/2, -\infty < z < h(x, y) \}.$$

Then we should find a wave profile $z = h(x, y)$ and the potential U such that

$$\Delta U = 0 \quad (\text{in } (x, y, z) \in \Omega_h), \quad (1.32)$$

$$\frac{\partial U}{\partial n} = 0 \quad (\text{on } z = h(x, y)), \quad (1.33)$$

$$\frac{1}{2}|\nabla U|^2 + gz + \frac{T}{m}K = \text{constant} \quad (\text{on } z = h(x, y)), \quad (1.34)$$

$$\lim_{z \rightarrow -\infty} \nabla U = (c, 0, 0), \quad (1.35)$$

where $\partial/\partial n$ denotes the outward normal derivative, $-c$ is the propagation speed, T is the surface tension coefficient, and K is the mean curvature of the free boundary. If we assume that the free boundary is represented as a nonparametric form $z = h(x, y)$, the mean curvature is given by

$$K = -\text{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right).$$

As in the two-dimensional case, z is assumed to be periodic in x and y .

Unlike in the two-dimensional case, we cannot use the streamfunction. Accordingly no transformation of the free boundary problem into a concise form like the Levi-Civita equation is known. Therefore numerical computation accompanies serious difficulties. See, for instance, [118, 121].

1.7 Remarks on equilibrium capillary surface

If we consider the case where the fluid is motionless, then we may set the velocity zero and we obtain an equation which describes equilibrium capillary surfaces. Recall the equation (1.10), which represents a balance among three terms: flow momentum, the gravity, and the capillary force. If the gravity is neglected, namely if $g = 0$, then we obtain the equation for the pure capillary waves, which will be discussed in Chapter 2. If the flow momentum can be neglected (this amount to assuming that g and T are

of the same order and both of them tend to infinity), then we obtain

$$gh(x, y) - \frac{T}{m} \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = \text{constant} \quad (1.36)$$

In the same way as in proving (1.18), we can prove that this equation has no solution other than $h \equiv \text{constant}$ (see the problem below), if we assume the periodicity on h . If h is sought with other nontrivial boundary conditions, then we obtain many interesting solutions.

For instance, let us consider the two-dimensional case, whence the free boundary (curve) is given by the following equation:

$$\alpha h - \left(\frac{h_x}{\sqrt{1 + h_x^2}} \right)_x = 0, \quad (1.37)$$

where $\alpha = mg/T$ is a positive constant. This equation can be integrated explicitly (see the problem below).

Another interesting case is the one where the periodic boundary condition is retained but g is negative. This is the case of fluid layer which is attached to the ceiling. In this case, (1.37) is essentially the same as the equation for Euler's elastica, see Chapter 6.

The equation is especially interesting if we consider the three dimensional case (hence two dimensional free surface). When g is negative, we obtain the following equation:

$$\alpha h + \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = 0,$$

where $\alpha = -gm/T > 0$.

We finally refer the reader to the best reference for equilibrium capillary surfaces, Finn [52].

1.8 Types of bifurcation

We now define some terminologies used in later chapters. Suppose that F maps $\mathbf{R} \times X$ into Y , where X and Y are Hilbert spaces or finite-dimensional spaces \mathbf{R}^N . What we have to do in the study of bifurcation phenomena is to look for zeros of F . $\{(\lambda, x) ; F(\lambda, x) = 0\}$ is typically a set of curves and we must know how curves intertwine among themselves.