of the same order and both of them tend to infinity), then we obtain

$$gh(x,y) - \frac{T}{m}\mathrm{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) = \text{constant} \tag{1.36}$$

In the same way as in proving (1.18), we can prove that this equation has no solution other than $h \equiv$ constant (see the problem below), if we assume the periodicity on h. If h is sought with other nontrivial boundary conditions, then we obtain many interesting solutions.

For instance, let us consider the two-dimensional case, whence the free boundary (curve) is given by the following equation:

$$\alpha h - \left(\frac{h_x}{\sqrt{1+h_x^2}}\right)_x = 0, \tag{1.37}$$

where $\alpha = mg/T$ is a positive constant. This equation can be integrated explicitly (see the problem below).

Another interesting case is the one where the periodic boundary condition is retained but g is negative. This is the case of fluid layer which is attached to the ceiling. In this case, (1.37) is essentially the same as the equation for Euler's elastica, see Chapter 6.

The equation is especially interesting if we consider the three dimensional case (hence two dimensional free surface). When g is negative, we obtain the following equation:

$$\alpha h + \mathrm{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) = 0,$$

where $\alpha = -gm/T > 0$.

We finally refer the reader to the best reference for equilibrium capillary surfaces, Finn [52].

1.8 Types of bifurcation

We now define some terminologies used in later chapters. Suppose that F maps $\mathbf{R}\times X$ into Y, where X and Y are Hilbert spaces or finite-dimensional spaces $\mathbf{R}^N$. What we have to do in the study of bifurcation phenomena is to look for zeros of F. $\{(\lambda, x)\ ;\ F(\lambda, x) = 0\ \}$ is typically a set of curves and we must know how curves intertwine among themselves.

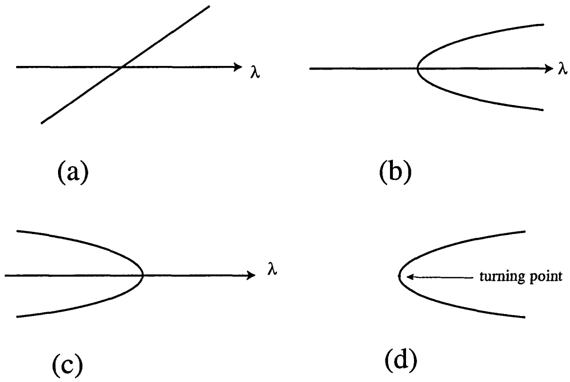

Fig. 1.3 Bifurcation diagram of (a) $F(\lambda, x) = \lambda x - ax^2$, (b) $F(\lambda, x) = \lambda x - bx^3$; $(0 < b)$, and (c) $F(\lambda, x) = \lambda x - bx^3$; $(b < 0)$. A turning point, (d).

Suppose now that $F(\lambda, 0) \equiv 0$. Then the curve $\{(\lambda, 0)\ ;\ \lambda \in \mathbf{R}\ \}$ is called the curve of trivial solutions. If another curve intersects it, the intersection is called the bifurcation point and the value of λ at the bifurcation point is called critical. If this is the case, we may have the following two possibilities; transcritical bifurcation and pitchfork bifurcation. A transcritical bifurcation is, by definition, a bifurcation where nontrivial solutions exists in both sides (with respect to λ) of the bifurcation point. On the other hand, a pitchfork bifurcation is the one where nontrivial solutions exists only in one side of the bifurcation point.

A typical example of a transcritical bifurcation is given by $F(\lambda, x) = \lambda x - ax^2$, where $x \in \mathbf{R}$ and a is a real constant (see Fig. 1.3 (a)). That of a pitchfork bifurcation is given by $F(\lambda, x) = \lambda x - bx^3$, where b is a constant. See Fig. 1.3, (b) and (c). The pitchfork bifurcation is classified further into supercritical bifurcation and subcritical bifurcation: if nontrivial solutions exist in a region where λ is larger than the critical value, then it is called a supercritical bifurcation (supercritical pitchfork); if they exist in a region where λ is smaller than the critical value, then it is called a subcritical bifurcation (subcritical pitchfork). Supercriticality and subcriticality are intimately connected with the stability of the nontrivial solutions if the dynamical system in question is a dissipative system. However, in the context of the conservative system like the one for the water-wave theory, stability or instability is not concluded by supercriticality or subcriticality alone.

Another terminology which frequently appears in the present book is a turning point. A turning point is defined as a point on the solution curve such that, in a small neighborhood of the point, the solutions exist in one side of the point and there exists no solution in the opposite side of the point. For instance, the origin is a turning point in the bifurcation diagram of $F(\lambda, x) = \lambda - x^2$. See Fig. 1.3, (d). A turning point is often called a limit point.

For more information about basic concepts in the bifurcation theory, see [36, 63, 64]. Very good applications can be found in [73].

1.9 Proof of Proposition 1.1

Proof. Let $\ell < m < n$ be integers and let (η, p, q) satisfy (1.27) for these three integers. Eliminating p and q from these three equations, we obtain

$$(\ell^2 - m^2)n(1 + r^n)(1 - r^m)(1 - r^\ell) + (n^2 - \ell^2)m(1 + r^m)(1 - r^n)(1 - r^\ell)$$
$$+(m^2 - n^2)\ell(1 + r^\ell)(1 - r^m)(1 - r^n) = 0,$$

where we have put $r = \eta^2$. Let $f(r)$ denote the left hand side. The proof ends if we have proven that $f(r) > 0$ for all $r \in [0, 1)$. Obviously $f(1) = 0$. Note also that

$$\begin{aligned} f(r) &= (m-\ell)(n-m)(n-\ell)(1+r^{n+m+\ell}) \\ &\quad -(m-\ell)(n+m)(n+\ell)(r^n + r^{m+\ell}) \\ &\quad +(n-\ell)(m+\ell)(m+n)(r^m + r^{n+\ell}) \\ &\quad -(n-m)(m+\ell)(n+\ell)(r^\ell + r^{n+m}). \end{aligned}$$

and $f(0) = (n - m)(n - \ell)(m - \ell) > 0$.

We put $f_1(r) = r^{1-\ell} f'(r)$, where the prime implies the differentiation. Then,

$$\begin{aligned} f_1(r) &= (m-\ell)(n-m)(n-\ell)(n+m+\ell)r^{n+m} \\ &\quad -(m-\ell)(n+m)(n+\ell)(nr^{n-\ell} + (m+\ell)r^m) \\ &\quad +(n-\ell)(m+\ell)(m+n)(mr^{m-\ell} + (n+\ell)r^n) \\ &\quad -(n-m)(m+\ell)(n+\ell)(\ell + (n+m)r^{n+m-\ell}). \end{aligned}$$

It satisfies $f_1(1) = 0$ and $f_1(0) < 0$. We inductively define functions $f_2 \cdots$