Another terminology which frequently appears in the present book is a turning point. A turning point is defined as a point on the solution curve such that, in a small neighborhood of the point, the solutions exist in one side of the point and there exists no solution in the opposite side of the point. For instance, the origin is a turning point in the bifurcation diagram of $F(\lambda, x) = \lambda - x^2$. See Fig. 1.3, (d). A turning point is often called a limit point.

For more information about basic concepts in the bifurcation theory, see [36, 63, 64]. Very good applications can be found in [73].

1.9 Proof of Proposition 1.1

Proof. Let $\ell < m < n$ be integers and let (η, p, q) satisfy (1.27) for these three integers. Eliminating p and q from these three equations, we obtain

$$(\ell^2 - m^2)n(1 + r^n)(1 - r^m)(1 - r^\ell) + (n^2 - \ell^2)m(1 + r^m)(1 - r^n)(1 - r^\ell)$$
$$+(m^2 - n^2)\ell(1 + r^\ell)(1 - r^m)(1 - r^n) = 0,$$

where we have put $r = \eta^2$. Let $f(r)$ denote the left hand side. The proof ends if we have proven that $f(r) > 0$ for all $r \in [0, 1)$. Obviously $f(1) = 0$. Note also that

$$\begin{aligned} f(r) &= (m-\ell)(n-m)(n-\ell)(1 + r^{n+m+\ell}) \\ &\quad -(m-\ell)(n+m)(n+\ell)(r^n + r^{m+\ell}) \\ &\quad +(n-\ell)(m+\ell)(m+n)(r^m + r^{n+\ell}) \\ &\quad -(n-m)(m+\ell)(n+\ell)(r^\ell + r^{n+m}). \end{aligned}$$

and $f(0) = (n-m)(n-\ell)(m-\ell) > 0$.

We put $f_1(r) = r^{1-\ell} f'(r)$, where the prime implies the differentiation. Then,

$$\begin{aligned} f_1(r) &= (m-\ell)(n-m)(n-\ell)(n+m+\ell)r^{n+m} \\ &\quad -(m-\ell)(n+m)(n+\ell)(nr^{n-\ell} + (m+\ell)r^m) \\ &\quad +(n-\ell)(m+\ell)(m+n)(mr^{m-\ell} + (n+\ell)r^n) \\ &\quad -(n-m)(m+\ell)(n+\ell)(\ell + (n+m)r^{n+m-\ell}). \end{aligned}$$

It satisfies $f_1(1) = 0$ and $f_1(0) < 0$. We inductively define functions $f_2 \cdots$

f_6 as follows. f_2 is defined as $f_2(r) = r^{1-m+\ell} f_1'(r)$. It is represented as follows:

$$\begin{aligned} f_2(r) \;=\;& (m-\ell)(n^2-m^2)(n-\ell)(n+m+\ell)r^{n+\ell} \\ & -(m-\ell)(n+m)(n+\ell)[n(n-\ell)r^{n-m} + m(m+\ell)r^{\ell}] \\ & +(n-\ell)(m+n)(m+\ell)[m(m-\ell) + n(n+\ell)r^{n-m+\ell}] \\ & -(n^2-m^2)(n+\ell)(m+\ell)(n+m-\ell)r^{n}. \end{aligned}$$

Definitions of f_3 through f_6 are slightly different depending on the sign of $n-m-\ell$. We first consider the case of $n-m-\ell \geq 0$. Then

$$\begin{aligned} f_3(r) \;\equiv\;& r^{1-\ell} f_2'(r) \\ =\;& (m-\ell)(n^2-m^2)(n^2-\ell^2)(n+m+\ell)r^{n} \\ & -(m-\ell)(n+m)(n+\ell)[n(n-\ell)(n-m)r^{n-m-\ell} + m(m+\ell)\ell] \\ & +n(n^2-\ell^2)(m+n)(m+\ell)(n-m+\ell)r^{n-m} \\ & -n(n^2-m^2)(n+\ell)(m+\ell)(n+m-\ell)r^{n-\ell}. \\ f_4(r) \;\equiv\;& r^{1-n+m+\ell} f_3'(r) \\ =\;& n(m-\ell)(n^2-m^2)(n^2-\ell^2)(n+m+\ell)r^{m+l} \\ & -n(m-\ell)(n^2-m^2)(n^2-\ell^2)(n-m-\ell) \\ & +n(n^2-m^2)(n^2-\ell^2)(m+\ell)(n-m+\ell)r^{\ell} \\ & -n(n^2-m^2)(n^2-\ell^2)(m+\ell)(n+m-\ell)r^{m}. \\ f_5(r) \;\equiv\;& r^{1-\ell} f_4'(r) \\ =\;& n(m^2-\ell^2)(n^2-m^2)(n^2-\ell^2)(n+m+\ell)r^{m} \\ & +n\ell(n^2-m^2)(n^2-\ell^2)(m+\ell)(n-m+\ell) \\ & -nm(n^2-m^2)(n^2-\ell^2)(m+\ell)(n+m-\ell)r^{m-\ell}. \\ f_6(r) \;\equiv\;& r^{1-m+\ell} f_5'(r) \\ =\;& nm(m^2-\ell^2)(n^2-m^2)(n^2-\ell^2)[(n+m+\ell)r^{\ell} - (n+m-\ell)]. \end{aligned}$$

Therefore f_6 is monotone increasing in $r \in [0,1)$ and satisfies $f_6(0) < 0$, $f_6(1) > 0$. This implies that there exists $r_1 \in (0,1)$ such that f_5 is decreasing in $0 < r < r_1$ and increasing in $r_1 < r < 1$. Since $f_5(0) > 0$, $f_5(1) = 0$, there is r_2 such that f_5 is positive in $0 < r < r_2$ and negative in $r_2 < r < 1$. This fact and

$$f_4(0) \begin{cases} < 0 & (n-m > \ell) \\ = 0 & (n-m = \ell) \end{cases}, \quad f_4(1) = 0$$

imply that the function f_4 must satisfy for some r_3,

$$f_4(r)\begin{cases} <0 & (0<r<r_3) \\ >0 & (r_3<r<1) \end{cases}, \quad \text{if } n-m>\ell,$$

and

$$f_4(r)>0 \quad (0<r<1), \quad \text{if } n-m=\ell.$$

In either case we have $f_3(r)<0$ for all $r\in(0,1)$, since $f_3(0)<0$ and $f_3(1)=0$. By virtue of $f_2(0)>0$ and $f_2(1)=0$, it holds that $f_2(r)>0$ for all $r\in(0,1)$. Since $f_1(0)<0$ and $f_1(1)=0$, the function f_1 is negative for all $r\in(0,1)$. This implies that f is monotone decreasing in r. Therefore, by $f(1)=0$, we have $f(r)>0$ for all $r\in[0,1)$.

The proof in the case of $n-m<\ell$ is similar. In this case we define as follows:

$$\begin{aligned}
f_3(r) &\equiv r^{1-n+m}f_2'(r) \\
&= (m-\ell)(n^2-m^2)(n^2-\ell^2)(n+m+\ell)r^{\ell+m} \\
&\quad -(m-\ell)(n+m)(n+\ell)[n(n-\ell)(n-m)+m(m+\ell)\ell r^{\ell+m-n}] \\
&\quad +n(n^2-\ell^2)(m+n)(m+\ell)(n-m+\ell)r^{\ell} \\
&\quad -n(n^2-m^2)(n+\ell)(m+\ell)(n+m-\ell)r^{m}. \\
f_4(r) &\equiv r^{1+n-m-\ell}f_3'(r) \\
&= (m^2-\ell^2)(n^2-m^2)(n^2-\ell^2)(n+m+\ell)r^{n} \\
&\quad -m\ell(m^2-\ell^2)(n+m)(n+\ell)(\ell+m-n) \\
&\quad +n\ell(n+m)(n^2-\ell^2)(m+\ell)(n-m+\ell)r^{n-m} \\
&\quad -nm(n^2-m^2)(n+\ell)(m+\ell)(n+m-\ell)r^{n-\ell}. \\
f_5(r) &\equiv r^{1-n+m}f_4'(r) \\
&= n(m^2-\ell^2)(n^2-m^2)(n^2-\ell^2)(n+m+\ell)r^{m} \\
&\quad +n\ell(n^2-m^2)(n^2-\ell^2)(m+\ell)(n-m+\ell) \\
&\quad -nm(n^2-m^2)(n^2-\ell^2)(m+\ell)(n+m-\ell)r^{m-\ell}. \\
f_6(r) &\equiv r^{1-m+\ell}f_5'(r) \\
&= nm(m^2-\ell^2)(n^2-m^2)(n^2-\ell^2)[(n+m+\ell)r^{\ell}-(n+m-\ell)].
\end{aligned}$$

As is seen, the expressions of f_5 and f_6 are the same as before. The proof proceeds similarly and we can conclude that $f(r)>0$ for any $r\in[0,1)$. □