Chapter 1

Vectors and Matrices

1.1 Introduction

In modeling and solving problems in engineering, economics, and in any other field of applied sciences there is often a need to present data in an organized way of a rectangular array. For example, if the prices of five items are listed as $(p_1, p_2, p_3, p_4, p_5)$ then such an array is constructed. This array has the specialty that it consists of only one row, therefore it is often called a *row vector*. Assume next, that a small firm produces three kinds of products. If the production levels are denoted by x_1, x_2, and x_3, then another type of special array

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

can be constructed, which consists of only one column. Therefore it is sometimes called a *column vector*. Consider again the same small firm and assume that for the next week the management considers two alternative production plans. Let x_1, x_2, x_3 and y_1, y_2, y_3 denote the alternative production volumes. The data can be conveniently summarized in a rectangular array form:

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}, \tag{1.1}$$

where the rows correspond to the different products, and the columns correspond to the alternative plans. This array consists of 3 rows and 2 columns, therefore it is usually called a *3 × 2 matrix* (pronounced "three by two" matrix). By constructing such arrays a new mathematical structure is developed.

Definition 1.1. For a given pair (m, n) of positive integers, an $m \times n$ *matrix* is a rectangular array of real (or complex) numbers given as

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

This matrix has m rows and n columns. Instead of saying that this matrix is $m \times n$ we may say that its *type* is $m \times n$. The numbers $a_{11}, a_{12}, \dots$ are called the *elements* or *entries* of the matrix. Notice, that each matrix element has two subscripts. The first subscript indicates the row in which the element is located, and the second subscript shows the column in which the element is placed. The set of all real (or complex) $m \times n$ matrices is denoted by $R^{m \times n}$ (or $C^{m \times n}$), which is the obvious generalization of the usual notation R (or C) for the set of all real (or complex) numbers.

Example 1.1. The type of matrix

$$\underline{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

is 2 ·3, since it consists of two rows and three columns, furthermore

$$a_{11} = 1,\ a_{12} = 2,\ a_{13} = 3,$$

$$a_{21} = 4,\ a_{22} = 5,\ a_{23} = 6.$$

♦

Matrices are usually denoted by underscore capital letters such as $\underline{A}, \underline{B}, \underline{C}$, and so on. Sometimes we refer to the matrix element a_{ij} as the (i, j) element or (i, j) entry of the matrix. In some applications it is convenient to use the notation $\underline{A} = (a_{ij})$, when a special emphasis is placed on the matrix elements. If one needs to indicate the type of matrix $\underline{A}$, then the simple notation $\underline{A}_{m \times n}$ or the slightly more complicated $\underline{A} = (a_{ij})_{m \times n}$ or

$$\underline{A} = \left(a_{ij}\right)_{i,j=1}^{m,\,n}$$

can be used, which shows that the value of i (the row-index) is between 1 and m, and the value of j (the column-index) is between 1 and n.

In most applications the rows and columns of matrices refer to certain quantities, parameters, or alternatives. If the prices of different products are summarized in a row vector (as it was done previously), then the columns refer to the different products, and if the production volumes are summarized in a column vector, then the rows refer to the products. Similarly, in the case of matrix (1.1), the rows correspond to the three products and the columns refer to the two production plans. In many cases it is useful to interchange the meanings of the rows and columns. Then a new matrix is constructed in which each column is formed from the elements of the corresponding row placed in the same order. The same result is obtained, when the elements of each column are placed in the corresponding row of the new matrix.

This matrix operation can be formally defined as follows.

Definition 1.2. Let $\underline{A}$ be an $m \times n$ matrix, then the *transpose* of $\underline{A}$ is the $n \times m$ matrix, the (i, j) element of which is a_{ji}. The transpose of $\underline{A}$ is denoted by $\underline{A}^T$, and this matrix operation is called *transposition.*

Example 1.2. The transpose of a row vector is a column vector:

$$(1,2,3)^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

the transpose of a column vector is a row vector:

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}^T = (4,5,6);$$

and the transpose of an $m \times n$ matrix is an $n \times m$ matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

♦

Notice that $\left(\underline{A}^T\right)^T = \underline{A}$, since the (i, j) element of $\underline{A}^T$ is a_{ji}, and therefore the (i, j) element of $\left(\underline{A}^T\right)^T$ is a_{ij}, which is the (i, j) element of the original matrix $\underline{A}$. In several cases it is convenient to emphasize in the notation if a matrix is a column vector or a row vector. Column vectors are denoted by underscore lower case letters such as $\underline{a}, \underline{b}, \underline{c}$, and so on. Since row vectors are the transposes of column vectors, they can be denoted as $\underline{a}^T, \underline{b}^T, \underline{c}^T$, and so on.

For an arbitrary matrix of the type $m \times n$, m of course, need not be equal to n. In the important special case of $m = n$, the matrix is called a *square matrix*. The common value of m and n is called the *order* of the matrix. The entries $a_{11}, a_{22}, \ldots, a_{mm}$ of a square matrix of order m are called the *diagonal elements*, and they form the main *diagonal* or simply the diagonal of the matrix.

Example 1.3. Matrix

$$\begin{pmatrix} 1 & 5 & 6 \\ 7 & 2 & 8 \\ 9 & 10 & 3 \end{pmatrix}$$

is a square matrix of order 3, and the elements 1, 2, and 3 form the diagonal of the matrix.

♦

A matrix composed entirely of zeros is called the *zero* (or *null*) *matrix*, and a vector of zeros is called a *zero* (or *null*) *vector*. A zero matrix is denoted by $\underline{O}$, and a zero column (or row) vector is denoted by $\underline{0}\left(\text{or } \underline{0}^T\right)$.

A square matrix with all off-diagonal elements equal to zero is a *diagonal* matrix. A special diagonal matrix, where all diagonal elements are equal to one, is called the *identity* matrix. The $n \times n$ identity matrix is usually denoted by $\underline{I}_n$. A square matrix in which all elements below the diagonal are equal to zero is called *upper triangular*, and similarly, a square matrix with all zero elements above the diagonal is called *lower triangular*.

Example 1.4. Consider matrices

$$\underline{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \underline{B} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \underline{C} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix},$$

then $\underline{A}$ is diagonal, $\underline{B}$ is upper triangular, and $\underline{C}$ is lower triangular. Notice that $\underline{A}$ (like any other diagonal matrix) satisfies the definitions of both upper and lower triangular matrices, therefore it is also an upper and a lower triangular.

♦

Notice that the transpose of an upper triangular matrix is lower triangular, and the transpose of a lower triangular matrix is upper triangular. It is easy to see that the transpose of a diagonal matrix is itself.

In economic theory, triangular matrices have a special meaning. Consider an $n \times n$ real square matrix $\underline{A}$, and assume that element a_{ij} represents the effect of unit i towards unit j, where the rows and columns of the matrix correspond to some economic units (for example, sectors in input–output models). In the case of an upper triangular matrix, $a_{ij}=0$ for $i>j$; and for lower triangular matrices $a_{ij}=0$ for $i<j$. That is, the zero matrix elements indicate that there is no effect from higher (or lower) indexed units to lower (or higher) indexed units showing a very strict hierarchy between the economic units.

Definition 1.3. An $n \times n$ matrix $\underline{A}$ is called *decomposable* if there is a nonempty proper subset J of $\{1,2,\dots,n\}$ such that

$$a_{ij}=0 \quad \text{for } i \notin J \text{ and } j \in J.$$

An $n \times n$ real matrix is called *indecomposable*, if it is not decomposable and is not the 1×1 zero matrix.

It is easy to see that a matrix is decomposable if and only if its transpose is decomposable. Any decomposable matrix can be transformed into the special form

$$\begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{O} & \underline{A}_{22} \end{pmatrix}$$

by interchanging its rows and columns, where $\underline{A}_{11}$ is $k \times k$, $\underline{A}_{12}$ is $k \times (n-k)$, $\underline{A}_{22}$ is an $(n-k) \times (n-k)$ matrix, $\underline{O}$ is the $(n-k) \times k$ zero matrix, and set J becomes $\{1,2,\dots,k\}$. In terms of the above economic interpretation the zero block indicates that there is no effect from the units not belonging to J towards the units of J.

Definition 1.4. A square matrix $\underline{A}$ is called *symmetric* if $\underline{A}^T = \underline{A}$, and it is called *skew-symmetric* if $\underline{A}^T = -\underline{A}$.

Notice that an $n \times n$ matrix is symmetric if and only if for $k = 1,2,\ldots,n$, its k^{th} columns have the same elements in the same order. As a special case, all diagonal matrices are symmetric. The diagonal of a skew symmetric matrix consists of zeros.

Example 1.5. Matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

is a symmetric 3 × 3 matrix.

♦

1.2 Comparison of Matrices

Matrices $\underline{A}$ and $\underline{B}$ are equal if they have the same type and the corresponding elements in the two matrices are equal. If $\underline{A} = (a_{ij})$ and $\underline{B} = (b_{ij})$, then $\underline{A} = \underline{B}$ if and only if $a_{ij} = b_{ij}$ for all i and j.

Similarly we say that for real matrices $\underline{A}$ and $\underline{B}$, $\underline{A} \le \underline{B}$, if they have the same type and for all i and j, $a_{ij} \le b_{ij}$. Analogously, $\underline{A} < \underline{B}$ if they have the same type and for all i and j, $a_{ij} < b_{ij}$.

Notice that matrices can be compared in the above sense only if they have the same type.

Example 1.6. Let

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \underline{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \underline{C} = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}, \underline{D} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix},$$

then for example, $\underline{A} = \underline{B}, \underline{A} \le \underline{C}, \underline{A} < \underline{D}$. If one defines

$$\underline{E} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

then it cannot be compared to any of matrices $\underline{A}, \underline{B}, \underline{C}$, or $\underline{D}$.

♦

In comparing matrices an important comment has to be made. If a and b are two real numbers, then exactly one of the relations $a = b, a < b$, and $a > b$ holds. That is, any two real numbers can be compared in this way. However real matrices may not be compared even if they have the same type. For example, row vectors $\underline{a}^T = (1,2)$ and $\underline{b}^T = (2,1)$ cannot be compared, since they are different, in the first element $\underline{a}^T$ is smaller but in the second element $\underline{b}^T$ is smaller. This phenomenon plays an important role in many fields of applied sciences. For example, in single objective optimization we are looking for a best solution, since any two values of the objective functions can be compared; however in the case of optimizing for multiple objectives we are looking for so-called efficient solutions, when none of the target function values can be improved without worsening another.

1.3 Elementary Matrix Algebra

In this section matrix operations will be introduced, and their main properties will be discussed.

Definition 1.5. Let $\underline{A}$ be an $m \times n$ real (or complex) matrix and let a be a real (or complex) number. The product $a\underline{A}$ is defined as the $m \times n$ matrix with (i, j) element $a \times a_{ij}$. That is, each element of the matrix is multiplied by a.

This definition can be briefly written as $a\underline{A} = (a \times a_{ij})_{m \times n}$.

As a simple example assume that the elements of a matrix represent cost data, and each element is given in dollars. If someone wants to change the dimension of the data to $1000, then each matrix element has to be multiplied by the same constant 0.001.

Example 1.7. For a numerical example assume that $a = 3$ and

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \text{ then } a\underline{A} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}.$$

♦

Notice, if $a = 0$, then $a\underline{A}$ is the zero matrix for all $\underline{A}$, since each matrix element is multiplied by zero. The real (or complex) number a is sometimes called a *scalar,* and this matrix operation is called the *multiplication by scalars.*

Definition 1.6. The *sum* of matrices $\underline{A}$ and $\underline{B}$ is defined whenever $\underline{A}$ and $\underline{B}$ have the same type. Each element of $\underline{A}+\underline{B}$ equals the sum of the two corresponding elements of $\underline{A}$ and $\underline{B}$. In other words, $\underline{A}+\underline{B}$ is the matrix the (i, j) element of which is $a_{ij} + b_{ij}$ for all i, j, where a_{ij} and b_{ij} are the (i, j) elements of $\underline{A}$ and $\underline{B}$, respectively. The *difference* matrix $\underline{A}-\underline{B}$ is analogously defined to be the matrix with (i, j) elements $a_{ij} - b_{ij}$.

We can summarize this definition as

$$\underline{A}+\underline{B} = \left(a_{ij} + b_{ij}\right)_{m\times n} \quad \text{and} \quad \underline{A}-\underline{B} = \left(a_{ij} - b_{ij}\right)_{m\times n}.$$

Example 1.8. Matrices

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \underline{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

cannot be added or subtracted, since they have different types. However

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

♦

The above matrix operations satisfy the following properties:

(a) If $\underline{A}$ and $\underline{B}$ have the same type, then

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}. \tag{1.2}$$

That is, matrix addition is *commutative*, which is a simple consequence of the fact that in adding matrices we add the corresponding matrix elements, and the addition of real (or complex) numbers is commutative.

(b) If $\underline{A}$, $\underline{B}$, and $\underline{C}$ have the same type, then

$$(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C}). \tag{1.3}$$

That is, matrix addition is *associative*. This property is also the simple consequence of the associativity of the addition of real (or complex) numbers. If $\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_k$ are real (or complex) matrices of the same type, then their sum is defined by the recursion $\underline{S}_1 = \underline{A}_1$ and $\underline{S}_i = \underline{S}_{i-1} + \underline{A}_i$ for $i = 2,3,\ldots,k$. Then $\underline{A}_1 + \underline{A}_2 + \ldots + \underline{A}_k = \underline{S}_k$.

(c) Let $\underline{A}$ be any matrix, and $\underline{O}$ be the zero matrix of the same order. Then

$$\underline{A} + \underline{O} = \underline{A}. \tag{1.4}$$

(d) If $\underline{A}$ and $\underline{B}$ have the same type, then with any scalar a,

$$a(\underline{A} + \underline{B}) = a\underline{A} + a\underline{B}; \tag{1.5}$$

and if a and b are two scalars, and $\underline{A}$ is any matrix, then

$$(a + b)\underline{A} = a\underline{A} + b\underline{A}. \tag{1.6}$$

These two equations are called the *distributivity* properties. Equation (1.5) shows distributivity with respect to the addition of matrices, and (1.6) is the distributive property with respect to the addition of scalars.

(e) If $\underline{A}$ and $\underline{B}$ have the same type, then

$$(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T. \tag{1.7}$$

Assume that both $\underline{A}$ and $\underline{B}$ are $m \times n$, then both sides of this equation are $n \times m$ matrices, and the (i, j) elements are the same: $a_{ji} + b_{ji}$ for all i and j.

(f) For positive integers k,

$$\underline{A} + \underline{A} + \ldots + \underline{A} = k\underline{A}, \tag{1.8}$$

where on the left-hand side we have k terms, each of them equals the same matrix $\underline{A}$.

(g) For all matrices $\underline{A}$ and $\underline{B}$ of the same type,

$$\underline{A} - \underline{B} = \underline{A} + (-1) \cdot \underline{B}, \tag{1.9}$$

since for all i and j, $a_{ij} - b_{ij} = a_{ij} + (-1) \cdot b_{ij}$.

Multiplication of matrices will be defined next. For simplifying the discussion, particular cases will be introduced before presenting the general definition.

As the first special case we define the product of row vectors by column vectors. Let $\underline{a}^T = (a_i)$ and $\underline{b} = (b_j)$ be a row and a column vector, respectively. The product $\underline{a}^T \underline{b}$ is defined only when the two vectors have the same number of elements, and in this case

$$\underline{a}^T \underline{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \sum_{i=1}^{n} a_i b_i,$$

where n is the common "length" of the vectors.

Example 1.9. The product

$$(1,2,3)\begin{pmatrix}4\\5\end{pmatrix}$$

cannot be defined, but

$$(1,2,3)\begin{pmatrix}2\\3\\4\end{pmatrix}=1\times 2+2\times 3+3\times 4=20.$$

♦

Notice that $\underline{a}^T\underline{b}$ is always a scalar, which can also be considered as a $1*1$ matrix. This multiplication can be illustrated and explained by the simple economic example, when a firm produces 3 items, the sale prices of which form the row vector $\underline{p}^T=(p_1,p_2,p_3)$, and the produced quantities are given in a column vector

$$\underline{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

Then the revenue (or total sale value) by selling the products is given by the product

$$\underline{p}^T\underline{x}=p_1x_1+p_2x_2+p_3x_3.$$

Assume next that $\underline{A}$ is an $m\times n$ matrix, and $\underline{x}$ is a column vector. The product $\underline{Ax}$ is defined only if the length of $\underline{x}$ equals the length of the rows of $\underline{A}$ (that is, when $\underline{x}$ has n elements), the product is an m-element column vector the i^{th} entry of which is obtained as the product of the i^{th} row of $\underline{A}$ by the column vector $\underline{x}$. That is, the i^{th} element of $\underline{Ax}$ equals $\sum_{j=1}^{n}a_{ij}x_j$.

Example 1.10. The product

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

cannot be defined, but

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}=\begin{pmatrix} 1\times 1+2\times 2 \\ 3\times 1+4\times 2 \end{pmatrix}=\begin{pmatrix} 5 \\ 11 \end{pmatrix}.$$

◆

Assume next that $\underline{x}^T$ is a row vector and $\underline{A}$ is an $m\times n$ matrix. The product $\underline{x}^T\ \underline{A}$ is defined only if the length of $\underline{x}^T$ equals the length of the columns of $\underline{A}$ (that is, when $\underline{x}$ has m-elements), the product is an n-element row vector, the $i^{\underline{\text{th}}}$ entry of which is obtained as the product of $\underline{x}^T$ by the $i^{\underline{\text{th}}}$ column of $\underline{A}$. That is, the $i^{\underline{\text{th}}}$ element of $\underline{x}^T\ \underline{A}$ equals

$$\sum_{j=1}^{m} x_j a_{ji}\ .$$

Example 1.11. The product

$$(1,2,3)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

cannot be defined, but

$$(1,2)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}=(1\times 1+2\times 3,\ 1\times 2+2\times 4)=(7,10)$$

◆

We are ready now to consider the general case of matrix multiplication.

Definition 1.7. Let $\underline{A}$ and $\underline{B}$ be two matrices. The product $\underline{AB}$ can be defined only if the rows of $\underline{A}$ have the same length as the columns of $\underline{B}$, and then the (i, j) element of $\underline{AB}$ equals the product of the $i^{\underline{\text{th}}}$ row of $\underline{A}$ by the $j^{\underline{\text{th}}}$ column of $\underline{B}$. If $\underline{A}$ is $m \times n$ and $\underline{B}$ is $p \times q$, then $\underline{AB}$ is defined only if $n = p$, its type is $m \times q$, and its (i, j) element is obtained as

$$\sum_{k=1}^{n} a_{ik} b_{kj} .$$

Example 1.12. The product

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

is not defined, but

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 11 & -11 \end{pmatrix},$$

since

$$(1,2)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \times 1 + 2 \times 2 = 5,$$

$$(1,2)\begin{pmatrix} -1 \\ -2 \end{pmatrix} = 1 \times (-1) + 2 \times (-2) = -5,$$

$$(3,4)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \times 1 + 4 \times 2 = 11,$$

and

$$(3,4)\begin{pmatrix}-1\\-2\end{pmatrix} = 3\times(-1)+4\times(-2) = -11.$$

♦

As we have seen before, a row vector can be multiplied by a column vector only if they have the same length, and the product is always a scalar. However a column vector can always be multiplied by a row vector even if they have different lengths, and the product is always a matrix, which is called a *dyad*. If $\underline{x}$ is an m-element column vector and $\underline{y}^T$ is an n-element row vector, then

$$\underline{x}\underline{y}^T = \begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}(y_1, y_2, \ldots, y_n) = \begin{pmatrix}x_1y_1 & x_1y_2 & \cdots & x_1y_n\\x_2y_1 & x_2y_2 & \cdots & x_2y_n\\\cdots & \cdots & \cdots & \cdots\\x_my_1 & x_my_2 & \cdots & x_my_n\end{pmatrix},$$

since the i^{th} row of $\underline{x}$ is the scalar x_i and the j^{th} column of $\underline{y}^T$ is the number y_j, and their product is x_iy_j.

If one selects two arbitrary matrices $\underline{A}$ and $\underline{B}$, then in their multiplication he/she should face one of the following possibilities:

(i) Neither $\underline{AB}$ nor $\underline{BA}$ exists. Such an example is provided by matrices

$$\underline{A} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} \text{ and } \underline{B} = \begin{pmatrix}1 & 2 & 3\\1 & 1 & 1\\2 & 1 & 0\end{pmatrix}.$$

(ii) Exactly one of the products $\underline{AB}$ and $\underline{BA}$ exists. For example, select

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } \underline{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

then

$$\underline{AB} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix},$$

but

$$\underline{BA} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

cannot be defined, since the rows of $\underline{B}$ have 3 elements and the columns of $\underline{A}$ have only 2.

(iii) Both $\underline{AB}$ and $\underline{BA}$ exist, but they have different types. As an example, select

$$\underline{A} = (1,1) \text{ and } \underline{B} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

then

$$\underline{AB} = (1,1)\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1 \times 2 + 1 \times 2 = 4$$

is a scalar and

$$\underline{BA} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}(1,1) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

is a dyad.

(iv) Both $\underline{AB}$ and $\underline{BA}$ exist, they have the same type but the products are not equal. Select

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } \underline{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$\underline{AB} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

and

$$\underline{BA} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This example also shows a different problem in matrix multiplication. In the second case $\underline{BA}$ is the zero matrix, however neither $\underline{A}$ nor $\underline{B}$ is zero, and furthermore there is no zero element in $\underline{A}$ or $\underline{B}$. This shows a different phenomenon from the one we used to have and apply in the case of real numbers. The product of real (or complex) numbers is zero only if at least one of the factors equals zero. This idea is used when one solves real equations by factorization. Unfortunately this method cannot be used in solving matrix equations.

(v) Both $\underline{AB}$ and $\underline{BA}$ exist, they have the same type and are equal. Such a special case can be illustrated by matrices

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \underline{B} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

when

$$\underline{AB} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

and

$$\underline{BA} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}.$$

Hence, the multiplication of matrices is not a commutative operation in general. However it satisfies the following properties:

(a) If the product $(\underline{AB}) \cdot \underline{C}$ exists, then $\underline{A} \cdot (\underline{BC})$ also exists and

$$(\underline{AB}) \cdot \underline{C} = \underline{A} \cdot (\underline{BC}). \tag{1.10}$$

That is, matrix multiplication is an *associative* operation. This property can be proved as follows. Introduce the notation $\underline{D} = \underline{AB}$ and $\underline{E} = \underline{BC}$, then the (i, j) element of $\underline{D}$ is given as

$$d_{ij} = \sum_l a_{il} b_{lj},$$

and therefore the (i, j) element of the left-hand side of (1.10) is as follows:

$$\sum_k d_{ik} c_{kj} = \sum_k \left(\sum_l a_{il} b_{lk} \right) c_{kj}.$$

On the other hand, the (i, j) element of $\underline{E}$ is

$$e_{ij} = \sum_k b_{ik} c_{kj},$$

and therefore the (i, j) element of the right-hand side of (1.10) equals

$$\sum_l a_{il} e_{lj} = \sum_l a_{il} \left(\sum_k b_{lk} c_{kj} \right).$$

Since a_{il} does not depend on the summation variable k, this expression gives the same value as the (i, j) element of the left-hand side of (1.10).

(b) If one of the matrices $(\underline{A} + \underline{B}) \cdot \underline{C}$ and $\underline{AC} + \underline{BC}$ exists, then the other matrix is also defined and they are equal:

$$(\underline{A} + \underline{B}) \cdot \underline{C} = \underline{AC} + \underline{BC}. \tag{1.11}$$

Similarly, if one of the matrices $\underline{A} \cdot (\underline{B} + \underline{C})$ and $\underline{AB} + \underline{AC}$ exists, then the other matrix is also defined and they are equal:

$$\underline{A} \cdot (\underline{B} + \underline{C}) = \underline{AB} + \underline{AC}. \tag{1.12}$$

These properties show that matrix multiplication is *distributive*, and their proof is similar to the one presented above for associativity.

(c) If $\underline{A}$ is $m \times n$, and $\underline{I}_m$ and $\underline{I}_n$ denote the $m \times m$ and $n \times n$ identity matrices, respectively, then

$$\underline{I}_m \cdot \underline{A} = \underline{A} \quad \text{and} \quad \underline{A} \cdot \underline{I}_n = \underline{A}. \tag{1.13}$$

These properties can also be proved in an easy way, the proofs are left as an exercise. Relations (1.13) show that multiplying by identity matrices leaves matrices unchanged. The same holds for real (or complex) numbers, when we multiply them by 1. Therefore identity matrices can be considered as the matrix-versions of the real number 1.

(d) If $\underline{AB}$ exists, then $\underline{B}^T \underline{A}^T$ also exists, furthermore

$$(\underline{AB})^T = \underline{B}^T \underline{A}^T. \tag{1.14}$$

A simple proof of this equation can be given by comparing the (i,j) elements of the two sides of the equality. The (i, j) element of $\underline{AB}$ equals $\sum_k a_{ik}b_{kj}$, therefore the (i,j) element of $(\underline{AB})^T$ is obtained by interchanging i and j: $\sum_k a_{jk}b_{ki} = \sum_k b_{ki}a_{jk}$.

This is the (i,j) element of the right-hand side, since b_{ki} is the (i,k) element of $\underline{B}^T$, and a_{jk} is the (k,j) element of $\underline{A}^T$.

(e) For any matrix $\underline{A}$ and zero matrix $\underline{O}$,

$$\underline{AO} = \underline{O} \quad \text{and} \quad \underline{OA} = \underline{O} \tag{1.15}$$

assuming that the left-hand sides are defined. Notice that if $\underline{A}$ is $m \times n$ and $\underline{O}$ is $n \times p$, then $\underline{A}\,\underline{O}$ is the $m \times p$ zero matrix, and if $\underline{A}$ is $m \times n$ and $\underline{O}$ is $p \times m$, then $\underline{O}\,\underline{A}$ is the $p \times n$ zero matrix.

Let $\underline{A}_1, \underline{A}_2, ..., \underline{A}_k$ be real (or complex) matrices of the types $m_1 \times n_1, m_2 \times n_2, ..., m_k \times n_k$, respectively, and assume that $n_1 = m_2$, $n_2 = m_3, ..., n_{k-1} = m_k$. The product of these matrices is defined by the recursion $\underline{P}_1 = \underline{A}_1$ and $\underline{P}_i = \underline{P}_{i-1}\underline{A}_i$ for $i = 2,3,...,k$ and letting $\underline{A}_1 \cdot \underline{A}_2 \cdot ... \cdot \underline{A}_k = \underline{P}_k$. In the special case, when $\underline{A}$ is an $n \times n$ square matrix we may select $\underline{A}_1 = \underline{A}_2 = ... = \underline{A}_k = \underline{A}$, and the product $\underline{A} \cdot \underline{A} \cdot ... \cdot \underline{A}$ can be simply denoted by $\underline{A}^k$. For convenience, we define $\underline{A}^0 = \underline{I}_n$ for all $n \times n$ square matrices $\underline{A}$.

Example 1.13. Select

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

then

$$\underline{A}^2 = \underline{A}\underline{A} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix},$$

$$\underline{A}^3 = \underline{A}^2\underline{A} = \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 9 & 18 \end{pmatrix},$$

and so on.

♦

Assume next that

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

is a single-variable polynomial with real (or complex) coefficients, and $\underline{A}$ is an $n \times n$ square matrix. The *matrix-polynomial* $p(\underline{A})$ is defined as

$$p(\underline{A}) = a_0 \underline{I}_n + a_1 \underline{A} + a_2 \underline{A}^2 + \ldots + a_m \underline{A}^m, \qquad (1.16)$$

where $\underline{I}_n$ is the $n \times n$ identity matrix.

Example 1.14. Let $p(x) = 2 + 2x + x^2$ and $\underline{A}$ as in the previous example, then

$$p(\underline{A}) = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 5 & 12 \end{pmatrix}.$$

♦

1.4 Inverse of a Matrix

We start this section with the definition of the inverse of a square matrix.

Definition 1.8. Let $\underline{A}$ be an $n \times n$ square matrix. The *inverse* of $\underline{A}$ is the $n \times n$ matrix $\underline{X}$ which satisfies the relation

$$\underline{A}\underline{X} = \underline{X}\underline{A} = \underline{I}_n, \tag{1.17}$$

where $\underline{I}_n$ is the $n \times n$ identity matrix. If such an $\underline{X}$ exists, then $\underline{A}$ is called an *invertible* matrix, and the inverse of $\underline{A}$ is denoted by $\underline{A}^{-1}$.

These relations show that inverse matrices generalize the concept of the reciprocal of a real (or complex) number, since for all real (or complex) $a \neq 0$,

$$aa^{-1} = a\frac{1}{a} = 1 \quad \text{and} \quad a^{-1}a = \frac{1}{a}a = 1.$$

If $a \neq 0$, then a^{-1} exists. Unfortunately, in the case of matrices the situation is more complicated, as it is illustrated in the following example.

Example 1.15. Consider matrix

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We will now prove that this matrix has no inverse. Assume that it has, and let

$$\underline{A}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then relation

$$\underline{AA}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}_2$$

implies that

$$\begin{aligned} a_{11} + a_{21} &= 1 \\ a_{12} + a_{22} &= 0 \\ a_{11} + a_{21} &= 0 \\ a_{12} + a_{22} &= 1, \end{aligned}$$

where we equated the (1,1), (1,2), (2,1), and (2,2) elements of the left-hand and right-hand sides, respectively. Notice that the first and third equations contradict each other, since $a_{11} + a_{21}$ must not have two different values at the same time. (Similar contradiction is obtained from the second and fourth equations.) In this case $\underline{A} \neq \underline{O}$, and the matrix even has no zero element. If we change the sign of only one element of $\underline{A}$ to get matrix

$$\underline{B} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

then

$$\underline{B}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which can be verified by simple calculation:

$$\underline{BB}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\underline{B}^{-1}\underline{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

♦

If $\underline{A}$ is the $n \times n$ zero matrix, then $\underline{AX} = \underline{O}$ for all $n \times n$ matrices $\underline{X}$, therefore $\underline{A}$ has no inverse. If $\underline{A} = \underline{I}_n$, then $\underline{A}^{-1} = \underline{I}_n$, since $\underline{I}_n \underline{I}_n = \underline{I}_n$.

At this moment we do not see an easy way to check if a given matrix has an inverse or not. In later chapters of this book we will introduce simple, practical conditions to check the existence of the inverse of a matrix. However we can easily show that in the case of the existence of an inverse of a given square matrix the inverse must be unique. Assume in contrary to the assertion that both $\underline{X}$ and $\underline{Y}$ are inverses of a matrix $\underline{A}$. Then

$$\underline{X} = \underline{X}\underline{I}_n = \underline{X}(\underline{AY}) = (\underline{X}\,\underline{A})\underline{Y} = \underline{I}_n\underline{Y} = \underline{Y};$$

hence $\underline{X}$ and $\underline{Y}$ are necessarily equal to each other. It is also easy to see that if both matrices $\underline{A}$ and $\underline{B}$ are invertible $n \times n$ matrices, then the inverse of their product also exists and

$$(\underline{AB})^{-1} = \underline{B}^{-1}\underline{A}^{-1}. \tag{1.18}$$

This relation can be verified by simple calculation:

$$(\underline{A}\underline{B})(\underline{B}^{-1}\underline{A}^{-1}) = \underline{A}(\underline{B}\underline{B}^{-1})\underline{A}^{-1} = (\underline{A}\underline{I})\underline{A}^{-1} = \underline{A}\underline{A}^{-1} = \underline{I},$$

and

$$(\underline{B}^{-1}\underline{A}^{-1})(\underline{A}\underline{B}) = \underline{B}^{-1}(\underline{A}^{-1}\underline{A})\underline{B} = (\underline{B}^{-1}\underline{I})\underline{B} = \underline{B}^{-1}\underline{B} = \underline{I}.$$

1.5 Further Examples and Applications

In this section some additional examples and applications of matrix algebra will be outlined.

1. Our first example is the *algebra of block matrices.* Assume that the $m \times n$ real (or complex) matrix $\underline{A}$ is divided into blocks as

$$\underline{A} = \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} & \cdots & \underline{A}_{1s} \\ \underline{A}_{21} & \underline{A}_{22} & \cdots & \underline{A}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{A}_{r1} & \underline{A}_{r2} & \cdots & \underline{A}_{rs} \end{pmatrix},$$

where $\underline{A}_{ij}$ is an $m_i \times n_j$ matrix. It is assumed that

$$m = \sum_{i=1}^{r} m_i \quad \text{and} \quad n = \sum_{j=1}^{s} n_j .$$

Suppose that matrix $\underline{B}$ has the same size as $\underline{A}$, and it is also divided into blocks as

$$\underline{B} = \begin{pmatrix} \underline{B}_{11} & \underline{B}_{12} & \cdots & \underline{B}_{1s} \\ \underline{B}_{21} & \underline{B}_{22} & \cdots & \underline{B}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{B}_{r1} & \underline{B}_{r2} & \cdots & \underline{B}_{rs} \end{pmatrix},$$

where for all i and j, blocks $\underline{A}_{ij}$ and $\underline{B}_{ij}$ have the same size. Since $\underline{A}$ and $\underline{B}$ are added and subtracted element-wise,

$$\underline{A}+\underline{B}=\begin{pmatrix} \underline{A}_{11}+\underline{B}_{11} & \underline{A}_{12}+\underline{B}_{12} & \cdots & \underline{A}_{1s}+\underline{B}_{1s} \\ \underline{A}_{21}+\underline{B}_{21} & \underline{A}_{22}+\underline{B}_{22} & \cdots & \underline{A}_{2s}+\underline{B}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{A}_{r1}+\underline{B}_{r1} & \underline{A}_{r2}+\underline{B}_{r2} & \cdots & \underline{A}_{rs}+\underline{B}_{rs} \end{pmatrix}, \quad (1.19)$$

and

$$\underline{A}-\underline{B}=\begin{pmatrix} \underline{A}_{11}-\underline{B}_{11} & \underline{A}_{12}-\underline{B}_{12} & \cdots & \underline{A}_{1s}-\underline{B}_{1s} \\ \underline{A}_{21}-\underline{B}_{21} & \underline{A}_{22}-\underline{B}_{22} & \cdots & \underline{A}_{2s}-\underline{B}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{A}_{r1}-\underline{B}_{r1} & \underline{A}_{r2}-\underline{B}_{r2} & \cdots & \underline{A}_{rs}-\underline{B}_{rs} \end{pmatrix}. \quad (1.20)$$

If the sizes of the corresponding blocks of $\underline{A}$ and $\underline{B}$ are different, we cannot add or subtract the corresponding blocks, since their sum and difference are defined only if they have the same size.

Example 1.16. Let

$$\underline{A}=\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \end{pmatrix}$$

be divided into four 2 × 2 blocks, and assume that

$$\underline{B}=\begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

is divided also into four blocks, but their sizes are 1 × 2, 1 × 2, 3 × 2, 3 × 2, respectively. Notice that both $\underline{A}+\underline{B}$ and $\underline{A}-\underline{B}$ exist, however neither of them can be obtained by adding or subtracting the corresponding blocks.

♦

Matrices can also be multiplied block-wise if the division of both matrices into blocks satisfy certain compatibility conditions. Assume that $\underline{A}$ is $m \times n$ and it is divided into blocks as before. Assume that $\underline{B}$ is an $n \times p$ matrix with block-form

$$\underline{B} = \begin{pmatrix} \underline{B}_{11} & \underline{B}_{12} & \cdots & \underline{B}_{1t} \\ \underline{B}_{21} & \underline{B}_{22} & \cdots & \underline{B}_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{B}_{s1} & \underline{B}_{s2} & \cdots & \underline{B}_{st} \end{pmatrix}$$

where the size of block $\underline{B}_{ij}$ is $n_i \times p_j$ with $p = \sum_{j=1}^{t} p_j$. For $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,t$ define

$$\underline{C}_{ij} = \sum_{l=1}^{s} \underline{A}_{il} \underline{B}_{lj}\,, \tag{1.21}$$

which is the "formal product" of the i^{th} block-row

$$\left(\underline{A}_{i1}, \underline{A}_{i2}, \ldots, \underline{A}_{is}\right)$$

of matrix $\underline{A}$ by the j^{th} block-column

$$\begin{pmatrix} \underline{B}_{1j} \\ \underline{B}_{2j} \\ \cdots \\ \underline{B}_{sj} \end{pmatrix}$$

of matrix $\underline{B}$. Then it is easy to see that the product $\underline{C} = \underline{A} \cdot \underline{B}$ can be divided into blocks as follows:

$$\underline{C} = \begin{pmatrix} \underline{C}_{11} & \underline{C}_{12} & \cdots & \underline{C}_{1t} \\ \underline{C}_{21} & \underline{C}_{22} & \cdots & \underline{C}_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{C}_{r1} & \underline{C}_{r2} & \cdots & \underline{C}_{rt} \end{pmatrix}.$$

Example 1.17. Select

$$\underline{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \underline{B} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

If both matrices are divided into 2 * 2 blocks as shown above, we have the blocks

$$\underline{A}_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \underline{A}_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \underline{A}_{21} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \underline{A}_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and

$$\underline{B}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \underline{B}_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \underline{B}_{21} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \underline{B}_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the 2×2 blocks of the product $\underline{C} = \underline{A} \cdot \underline{B}$ can be obtained as follows:

$$\underline{C}_{11}=\underline{A}_{11}\underline{B}_{11}+\underline{A}_{12}\underline{B}_{21}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}$$
$$=\begin{pmatrix}1&1\\1&2\end{pmatrix}+\begin{pmatrix}2&2\\0&0\end{pmatrix}=\begin{pmatrix}3&3\\1&2\end{pmatrix},$$

$$\underline{C}_{12}=\underline{A}_{11}\underline{B}_{12}+\underline{A}_{12}\underline{B}_{22}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$$
$$=\begin{pmatrix}2&2\\3&3\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}3&2\\3&3\end{pmatrix},$$

$$\underline{C}_{21}=\underline{A}_{21}\underline{B}_{11}+\underline{A}_{22}\underline{B}_{21}=\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}$$
$$=\begin{pmatrix}1&1\\0&0\end{pmatrix}+\begin{pmatrix}2&2\\3&3\end{pmatrix}=\begin{pmatrix}3&3\\3&3\end{pmatrix},$$

and

$$\underline{C}_{22}=\underline{A}_{21}\underline{B}_{12}+\underline{A}_{22}\underline{B}_{22}=\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$$
$$=\begin{pmatrix}2&2\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}3&2\\1&0\end{pmatrix}.$$

Therefore,

$$\underline{A}\cdot\underline{B}=\underline{C}=\begin{pmatrix}3&3&3&2\\1&2&3&3\\3&3&3&2\\3&3&1&0\end{pmatrix}.$$

The direct multiplication of matrices $\underline{A}$ and $\underline{B}$ has to give the same answer, that can be easily checked.

♦

If the division of $\underline{A}$ and $\underline{B}$ into blocks does not satisfy the above conditions then $\underline{A}$ and $\underline{B}$ cannot be multiplied by using blocks even in cases when $\underline{A} \cdot \underline{B}$ exists. In such cases the original definition of matrix multiplication by using the matrix elements can be used.

As the conclusion of this example we will examine inverses of block matrices. Assume that the $n \times n$ matrix $\underline{A}$ is divided into blocks as

$$\underline{A} = \begin{pmatrix} \underline{P} & \underline{Q} \\ \underline{R} & \underline{S} \end{pmatrix},$$

where $\underline{P}$ is $m \times m$, $\underline{Q}$ is $m \times (n-m)$, $\underline{R}$ is $(n-m) \times m$, and $\underline{S}$ is an $(n-m) \times (n-m)$ matrix. We will determine the inverse of $\underline{A}$ in a similar block-form

$$\underline{A}^{-1} = \begin{pmatrix} \underline{X} & \underline{Y} \\ \underline{U} & \underline{V} \end{pmatrix},$$

where $\underline{X}$ is $m \times m$, $\underline{Y}$ is $m \times (n-m)$, $\underline{U}$ is $(n-m) \times m$, and $\underline{V}$ is $(n-m) \times (n-m)$. The definition of inverse matrices implies that

$$\begin{pmatrix} \underline{P} & \underline{Q} \\ \underline{R} & \underline{S} \end{pmatrix} \begin{pmatrix} \underline{X} & \underline{Y} \\ \underline{U} & \underline{V} \end{pmatrix} = \begin{pmatrix} \underline{I}_m & \underline{O} \\ \underline{O} & \underline{I}_{n-m} \end{pmatrix}.$$

Comparing the corresponding blocks of both sides of this equation gives the relations

$$\begin{aligned} \underline{P}\underline{X} + \underline{Q}\underline{U} &= \underline{I}_m \\ \underline{P}\underline{Y} + \underline{Q}\underline{V} &= \underline{O} \\ \underline{R}\underline{X} + \underline{S}\underline{U} &= \underline{O} \\ \underline{R}\underline{Y} + \underline{S}\underline{V} &= \underline{I}_{n-m}. \end{aligned} \tag{1.22}$$

Assuming that $\underline{P}$ is invertible, the second equation implies that

$$\underline{Y} = -\underline{P}^{-1}\underline{Q}\underline{V}, \tag{1.23}$$

and substituting this relation into the fourth equation gives an equation for block $\underline{V}$:

$$\left(-\underline{R}\underline{P}^{-1}\underline{Q} + \underline{S}\right)\underline{V} = \underline{I}_{n-m},$$

that is, $\underline{S} - \underline{R}\underline{P}^{-1}\underline{Q}$ must be invertible, and

$$\underline{V} = \left(\underline{S} - \underline{R}\underline{P}^{-1}\underline{Q}\right)^{-1}. \tag{1.24}$$

From the third equation of (1.22) we have

$$\underline{U} = -\underline{S}^{-1}\underline{R}\underline{X} \tag{1.25}$$

assuming that $\underline{S}$ is invertible. Substitute this equation into the first equation of (1.22) to see that $\underline{P} - \underline{Q}\underline{S}^{-1}\underline{R}$ must be invertible, and

$$\underline{X} = \left(\underline{P} - \underline{Q}\underline{S}^{-1}\underline{R}\right)^{-1}. \tag{1.26}$$

Notice that equations (1.26), (1.25), (1.24), (1.23) can be used to recover the unknown blocks $\underline{X}$, $\underline{U}$, $\underline{V}$, and $\underline{Y}$ of the inverse matrix A^{-1}.

Example 1.18. We will now invert matrix

$$\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In this case we may select

$$\underline{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \underline{Q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \underline{R} = (1,1), \quad \text{and} \quad \underline{S} = (1).$$

Equation (1.26) implies that

$$\underline{X} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 1 \cdot (1,1) \right)^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

since $\underline{S}^{-1} = (1)$. From equation (1.25) we have

$$\underline{U} = -1 \cdot (1,1) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = (1,1).$$

Equation (1.24) is then applied to find $\underline{V}$:

$$\underline{V} = \left(1 - (1,1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^{-1} = (1-2)^{-1} = \frac{1}{-1} = -1,$$

where we used the fact that the inverse of the identity matrix is itself. And finally, from equation (1.23) we get

$$\underline{Y} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot (-1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence,

$$\underline{A}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

♦

In many applications a *certain part of a matrix* (a row, column, element, or even a block of the matrix) is needed for further computation. In this example of matrix algebra we will show how to obtain such matrix parts by using only matrix operations. Let $\underline{A}$ be a given $m \times n$ real (or complex) matrix with (i, j) element a_{ij}.

Let $\underline{e}_j^{(n)}$ denote the n-element column vector the $j^{\underline{\text{th}}}$ element of which is one, and all other elements are equal to zero. Then

$$\underline{A} \cdot \underline{e}_j^{(n)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow j^{\underline{\text{th}}} \text{ element} \\ \\ \\ \end{matrix}$$

$\uparrow$

$j^{\underline{\text{th}}}$ column

For $i = 1, 2, \ldots, m$, the $i^{\underline{\text{th}}}$ element of the product is obtained by multiplying the $i^{\underline{\text{th}}}$ row of $\underline{A}$ by the column vector $\underline{e}_j^{(n)}$:

$$(a_{i1}, a_{i2}, \ldots, a_{ij}, \ldots, a_{in}) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\underline{\text{th}}} \text{ element} = a_{ij},$$

$$\uparrow$$
$$j^{\underline{\text{th}}} \text{ element}$$

since all other terms equal zero. Hence

$$\underline{A} \cdot \underline{e}_j^{(n)} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix},$$

which is the $j^{\underline{\text{th}}}$ column of A.

Let now $\underline{e}_i^{(m)T}$ denote the m-element row vector the $i^{\underline{\text{th}}}$ element of which is equal to one and all other elements are zeros. Then

$$\underline{e}_i^{(m)T} \underline{A} = (0,0,\ldots,1,\ldots 0) \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \leftarrow i^{\underline{\text{th}}} \text{ row.}$$

$$\uparrow$$
$$i^{\underline{\text{th}}} \text{ element}$$

For $j = 1,2,\dots,n$, the $j^{\underline{\text{th}}}$ element of the product is obtained by multiplying $\underline{e}_i^{(m)T}$ by the $j^{\underline{\text{th}}}$ column of $\underline{A}$:

$$(0,0,\dots,1,\dots,0)\begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{in} \end{pmatrix} \leftarrow i^{\underline{\text{th}}} \text{ element } = a_{ij},$$

$\uparrow$

$i^{\underline{\text{th}}}$ element

since all other terms are equal to zero. Therefore

$$\underline{e}_i^{(m)T}\underline{A} = (a_{i1}, a_{i2}, \dots, a_{ij}, \dots, a_{in}),$$

which is the $i^{\underline{\text{th}}}$ row of $\underline{A}$.

It is also easy to see that for all i and j,

$$a_{ij} = \underline{e}_i^{(m)T}\underline{A}\underline{e}_j^{(n)},$$

since

$$\underline{e}_i^{(m)T}\left(\underline{A}\underline{e}_j^{(n)}\right) = (0,0,\dots,1,\dots,0)\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} \leftarrow i^{\underline{\text{th}}} \text{ element } = a_{ij},$$

$\uparrow$

$i^{\underline{\text{th}}}$ element

since all other terms equal zero.

The above relations can be presented in a much more general framework. Let $1 \le i_1 < i_2 < ... < i_r \le m$ and $1 \le j_1 < j_2 < ... < j_s \le n$ be arbitrary integers. Consider matrix

$$\underline{A}_1 = \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_s} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_r j_1} & a_{i_r j_2} & \cdots & a_{i_r j_s} \end{pmatrix}$$

which can be obtained from $\underline{A}$ by deleting all rows except rows $i_1, i_2, ..., i_r$ and all columns except columns $j_1, j_2, ..., j_s$. Define the $r \times m$ matrix $\underline{U}$ and the $n \times s$ matrix $\underline{V}$ such that

$$u_{1i_1} = u_{2i_2} = ... = u_{ri_r} = 1, \quad \text{all other} \quad u_{ij} = 0;$$

and

$$v_{j_1 1} = v_{j_2 2} = ... = v_{j_s s} = 1, \quad \text{all other} \quad v_{ij} = 0.$$

Then

$$\underline{A}_1 = \underline{U}\underline{A}\underline{V}. \tag{1.27}$$

In the particular case, when $\underline{A}$ is a square matrix, $r = s$, and $i_1 = j_1, i_2 = j_2, ..., i_r = j_r$, then matrix $\underline{A}_1$ called a *principal submatrix* of $\underline{A}$. Notice that all principal submatrices are square matrices, and if $\underline{A}$ is $n \times n$, then there are $\binom{n}{r}$ $r \times r$ principal submatrices of $\underline{A}$.

Using a similar idea as before, some further vector characteristics can be derived, that have significant applications in statistics.

Let $x_1, ..., x_n$ be sample elements. They can be summarized as a column vector

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

By introducing the n-element row vector

$$\underline{1}^T = (1,1,\ldots,1)$$

it is easy to see that

$$\underline{1}^T \underline{x} = x_1 + x_2 + \ldots + x_n,$$

therefore the sample mean can be obtained as

$$\bar{x} = \frac{1}{n} \underline{1}^T \underline{x}.$$

If we notice that $\underline{1}^T \underline{1} = n$, then we also have

$$\bar{x} = \frac{\underline{1}^T \underline{x}}{\underline{1}^T \underline{1}}.$$

The sample variance can be expressed as

$$S^2 = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x})^2.$$

Notice now that

$$\underline{x} - \bar{x} \cdot \underline{1} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \cdots \\ x_n - \bar{x} \end{pmatrix},$$

therefore

$$S_x^2 = \frac{1}{n-1}\left(\underline{x} - \bar{x} \cdot \underline{1}\right)^T \left(\underline{x} - \bar{x} \cdot \underline{1}\right) = \frac{\left(\underline{x} - \bar{x} \cdot \underline{1}\right)^T \left(\underline{x} - \bar{x} \cdot \underline{1}\right)}{n-1}$$

$$= \frac{\left(\underline{x} - \frac{\underline{1}^T \underline{x}}{\underline{1}^T \underline{1}} \cdot \underline{1}\right)^T \left(\underline{x} - \frac{\underline{1}^T \underline{x}}{\underline{1}^T \underline{1}} \cdot \underline{1}\right)}{\underline{1}^T \underline{1} - 1}.$$

Consider next two n-element samples, $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. The covariance between these samples can be written as follows:

$$Cov\left(\underline{x}, \underline{y}\right) = \frac{1}{n}\sum_{k=1}^{n}\left(x_k - \bar{x}\right)\left(y_k - \bar{y}\right) = \frac{1}{n}\left(\underline{x} - \bar{x} \cdot \underline{1}\right)^T \left(\underline{y} - \bar{y} \cdot \underline{1}\right)$$

$$= \frac{\left(\underline{x} - \frac{\underline{1}^T \underline{x}}{\underline{1}^T \underline{1}}\right)\left(\underline{y} - \frac{\underline{1}^T \underline{y}}{\underline{1}^T \underline{1}}\right)}{\underline{1}^T \underline{1}}.$$

The correlation between the two samples has the general form

$$r = \frac{Cov\left(\underline{x}, \underline{y}\right)}{S_x \cdot S_y},$$

it can also be expressed by using vector operations if we substitute the above expressions for the covariance and the two variances.

Time invariant linear *dynamic economic systems with discrete* time scale can be generally formulated as the difference equation

$$\underline{x}(t+1) = \underline{A}\underline{x}(t) + \underline{b}, \tag{1.28}$$

where the n-element vector $\underline{x}$ is the state variable, $\underline{A}$ is a given $n \times n$ real matrix, and $\underline{b}$ is a given n-element real vector. The initial state $\underline{x}_0 = \underline{x}(0)$ is also assumed to be known. An elementary problem of systems theory is to find $\underline{x}(t)$ for all future times as easily as possible. In this application we will suggest a solution for this problem. Substitute t=0,1 and 2 into equation (1.28) to see that

$$\begin{aligned}
\underline{x}(1) &= \underline{A}\underline{x}(0)+\underline{b} = \underline{A}\underline{x}_0+\underline{b},\\
\underline{x}(2) &= \underline{A}\underline{x}(1)+\underline{b} = \underline{A}(\underline{A}\underline{x}_0+\underline{b})+\underline{b} = \underline{A}^2\underline{x}_0+(\underline{A}+\underline{I})\underline{b},\\
\underline{x}(3) &= \underline{A}\underline{x}(2)+\underline{b} = \underline{A}(\underline{A}^2\underline{x}_0+(\underline{A}+\underline{I})\underline{b})+\underline{b} = \underline{A}^3\underline{x}_0+(\underline{A}^2+\underline{A}+\underline{I})\underline{b}.
\end{aligned}$$

These initial solution vectors suggest that in general,

$$\underline{x}(t) = \underline{A}^t\underline{x}_0+(\underline{A}^{t-1}+\underline{A}^{t-2}+\ldots+\underline{A}+\underline{I})\underline{b} = \underline{A}^t\underline{x}_0+\left(\sum_{l=0}^{t-1}\underline{A}^l\right)\underline{b} \qquad (1.29)$$

where we use the fact that $\underline{I}=\underline{A}^0$. This solution formula can be proved by finite induction. For $t=1,2,$ and 3 the formula is valid as it is shown from the above initial values of the state variable. Assume that the formula is valid for an integer $t>0$. Then from equation (1.28) we conclude that

$$\underline{x}(t+1) = \underline{A}\underline{x}(t)+\underline{b} = \underline{A}\left(\underline{A}^t\underline{x}_0+\left(\sum_{l=0}^{t-1}\underline{A}^l\right)\underline{b}\right)+\underline{b} = \underline{A}^{t+1}\underline{x}_0+\left(\sum_{l=0}^{t}\underline{A}^l\right)\underline{b},$$

that is, the formula remains valid for $t+1$. Hence it holds for all $t\geq 1$.

In applying the solution formula (1.29), we need a fast method to find powers of $\underline{A}$. Later, in Chapter 6 we will show a general method for the efficient computation of $\underline{A}^t$ for all $t\geq 1$. In many special cases the power matrix $\underline{A}^t$ can be determined by calculating some initial powers $\underline{A}^2, \underline{A}^3, \underline{A}^4$, and observing the general formula, which has to be then proved by finite induction.

Example 1.19. Consider matrix

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Then

$$\underline{A}^2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix} = 3 \cdot \underline{A},$$

$$\underline{A}^3 = \underline{A}^2 \cdot \underline{A} = (3 \cdot \underline{A})\underline{A} = 3 \cdot \underline{A}^2 = 3(3 \cdot \underline{A}) = 3^2 \cdot \underline{A},$$

$$\underline{A}^4 = \underline{A}^3 \cdot \underline{A} = (3^2 \cdot \underline{A})\underline{A} = 3^2 \cdot \underline{A}^2 = 3^2(3 \cdot \underline{A}) = 3^3 \cdot \underline{A}.$$

By using finite induction it is easy to show that in general,

$$\underline{A}^t = 3^{t-1}\underline{A}.$$

We will next solve the difference equation

$$\underline{x}(t+1) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\underline{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \underline{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From equation (1.29) we have

$$\underline{x}(t) = 3^{t-1}\underline{A}\underline{x}_0 + \left(\underline{I} + \left(\sum_{l=1}^{t-1} 3^{l-1}\right)\underline{A}\right)\underline{b} = 3^{t-1}\underline{A}\underline{x}_0 + \left(\underline{I} + \frac{3^{t-1}-1}{3-1}\underline{A}\right)\underline{b}.$$

Notice that for $\underline{A}^0$ we must not use the general formula of $\underline{A}^t$ since it holds usually only for $t \geq 1$. We have to use the fact that $\underline{A}^0 = \underline{I}$. Substituting the actual form of $\underline{A}$, $\underline{x}_0$, and $\underline{b}$ into the above equation gives the solution:

$$\underline{x}(t)=3^{t-1}\begin{pmatrix}1&1\\2&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}+\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{1}{2}\left(3^{t-1}-1\right)\begin{pmatrix}1&1\\2&2\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$=3^{t-1}\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}+\frac{3^{t-1}-1}{2}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}\frac{5}{2}\cdot 3^{t-1}+\frac{1}{2}\\ 5\cdot 3^{t-1}-1\end{pmatrix}.$$

◆

Time invariant linear *dynamic economic systems with continuous time scale* can be generally formulated as the differential equation

$$\dot{\underline{x}}(t)=\underline{A}\cdot\underline{x}(t)+\underline{b} \tag{1.30}$$

where $\underline{x}$ is the state variable, matrix $\underline{A}$ is $n\times n$, and vector $\underline{b}$ has n elements. It is well known from linear systems theory (see, for example, Szidarovszky and Bahill, 1992) that the solution is given as

$$\underline{x}(t)=e^{\underline{A}\cdot t}\underline{x}_0+\int_0^t e^{\underline{A}\cdot(t-\tau)}\underline{b}d\tau. \tag{1.31}$$

In Chapter 6 we will show a general method to compute matrix exponentials, which are defined as the sum of the infinite series

$$e^{\underline{A}t}=\underline{I}+\frac{\underline{A}\cdot t}{1!}+\frac{\underline{A}^2t^2}{2!}+\frac{\underline{A}^3t^3}{3!}+\dots\quad . \tag{1.32}$$

It is also well known that this series is convergent for all real t and arbitrary $n\times n$ real matrices. In special cases, a general formula may be derived for $\underline{A}^k$, and this general formula can be substituted into equality (1.32) to get a closed form for $e^{\underline{A}t}$.

Example 1.20. Consider again the 2 * 2 matrix

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

In Example 1.19 we have seen that for all $k \geq 1$,

$$\underline{A}^k = 3^{k-1} \cdot \underline{A} = 3^{k-1} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Therefore

$$\begin{aligned} e^{\underline{A}t} &= \underline{I} + \sum_{k=1}^{\infty} \frac{3^{k-1} \underline{A} t^k}{k!} = \underline{I} + \underline{A} \cdot \sum_{k=1}^{\infty} \frac{3^{k-1} t^k}{k!} \\ &= \underline{I} + \underline{A} \cdot \frac{1}{3} \sum_{k=1}^{\infty} \frac{(3t)^k}{k!} = \underline{I} + \underline{A} \cdot \frac{1}{3}\left(e^{3t} - t\right) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - 1}{3} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\left(e^{3t} + 2\right) & \frac{1}{3}\left(e^{3t} - 1\right) \\ \frac{2}{3}\left(e^{3t} - 1\right) & \frac{1}{3}\left(2e^{3t} + 1\right) \end{pmatrix}. \end{aligned}$$

We will next solve the initial value-problem

$$\underline{\dot{x}}(t) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \underline{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, we will derive a formula for $\underline{x}(t)$ at all future times $t > 0$ by assuming that the initial state at $t = 0$ is given. From equation (1.31) we have

$$\underline{x}(t) = \begin{pmatrix} \frac{1}{3}\left(e^{3t}+2\right) & \frac{1}{3}\left(e^{3t}-1\right) \\ \frac{2}{3}\left(e^{3t}-1\right) & \frac{1}{3}\left(2e^{3t}+1\right) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$+ \int_0^t \begin{pmatrix} \frac{1}{3}\left(e^{3(t-\tau)}+2\right) & \frac{1}{3}\left(e^{3(t-\tau)}-1\right) \\ \frac{2}{3}\left(e^{3(t-\tau)}-1\right) & \frac{1}{3}\left(2e^{3(t-\tau)}+1\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} \frac{1}{3}\left(2e^{3t}+1\right) \\ \frac{1}{3}\left(4e^{3t}-1\right) \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{3}\left(e^{3(t-\tau)}+2\right) \\ \frac{2}{3}\left(e^{3(t-\tau)}-1\right) \end{pmatrix} d\tau.$$

Notice that

$$\int_0^t \frac{1}{3}\left(e^{3(t-\tau)}+2\right) d\tau = \frac{1}{3}\left[\frac{e^{3(t-\tau)}}{-3} + 2\tau\right]_{\tau=0}^{t}$$

$$= \frac{1}{3}\left[\left(-\frac{1}{3}+2t\right) - \left(-\frac{1}{3}e^{3t}\right)\right] = \frac{1}{9}\left(e^{3t}+6t-1\right)$$

and

$$\int_0^t \frac{2}{3}\left(e^{3(t-\tau)}-1\right) d\tau = \frac{2}{3}\left[\frac{e^{3(t-\tau)}}{-3} - \tau\right]_{\tau=0}^{t}$$

$$= \frac{2}{3}\left[\left(-\frac{1}{3}-t\right) - \left(-\frac{1}{3}e^{3t}\right)\right] = \frac{2}{9}\left(e^{3t}-3t-1\right),$$

therefore

$$\underline{x}(t) = \begin{pmatrix} \frac{1}{9}\left(7e^{3t} + 6t + 2\right) \\ \frac{1}{9}\left(14e^{3t} - 6t - 5\right) \end{pmatrix}.$$

♦

A special linear dynamic system will be introduced next, which is called the *dynamic Cournot oligopoly model.* Assume that n firms produce a homogeneous good and sell it on the same market. Assume that the market demand function $d(p)$ is decreasing and linear, where p is the selling price. Then its inverse is also decreasing and linear:

$$p(d) = a \cdot d + b \quad (a < 0,\, b > 0).$$

The constant b shows the high price that the consumers would be willing to pay if the product is not available on the market directly. The coefficient a shows the decrease in the price if the quantity of available products increases by a unit. Assume that the production cost of firm $k\,(1 \le k \le n)$ is also linear:

$$C_k(x_k) = b_k x_k + c_k \quad (b_k > 0, c_k \ge 0),$$

where x_k denotes the production level of firm k. Here c_k shows the fixed cost, and b_k is the marginal cost (that is, the additional cost arising by the increase of the production level by a unit). The profit of firm k is therefore given as

$$\varphi_k(x_1,\ldots,x_n) = x_k\left(a\left(\sum_{l=1}^{n} x_l\right) + b\right) - (b_k x_k + c_k).$$

Consider next the following *discrete dynamic extension* of this model. Let $x_1(0),\ldots,x_n(0)$ denote the production levels of the firms at the initial time period $t = 0$. Assume that at each further time period $t + 1(t \ge 0)$, each firm maximizes its profit under the assumption that the competitors do not change their production levels from the previous time period. That is, firm k $(k=1,2,\dots,n)$ maximizes its profit

$$x_k\left(a\left(x_k+\sum_{l\neq k}x_l(t)\right)+b\right)-(b_kx_k+c_k).$$

Assuming interior optimum, simple differentiation shows that the optimal solution for x_k is given as

$$x_k(t+1)=-\frac{1}{2}\sum_{l\neq k}x_l(t)+\frac{b_k-b}{2a}. \tag{1.33}$$

This equation is the special case of the difference equation (1.28) with

$$\underline{A}=\begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \underline{b}=\begin{pmatrix} \frac{b_1-b}{2a} \\ \frac{b_2-b}{2a} \\ \cdots \\ \frac{b_n-b}{2a} \end{pmatrix}.$$

The *continuous dynamic extension* of the model can be formulated in the following way. Assume again that $x_1(0),x_2(0),\dots,x_n(0)$ are the initial production levels. At each time period $t\geq 0$, each firm adjusts its production level proportionally to its marginal profit, that is, for all k,

$$\dot{x}_k(t)=m_k\cdot\left(2ax_k(t)+a\sum_{l\neq k}x_l(t)+b-b_k\right), \tag{1.34}$$

where $m_k > 0$ is a given constant. Notice that the marginal profit of firm k is the derivative of his profit with respect to x_k, and equation (1.34) requires that x_k increases if the profit increases in x_k, and x_k decreases if the profit decreases in x_k. These equations can be summarized as the differential equation (1.30) with

$$\underline{A} = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix} \begin{pmatrix} 2a & a & \dots & a & a \\ a & 2a & \dots & a & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & \dots & a & 2a \end{pmatrix} \quad \text{and}$$

$$\underline{b} = \begin{pmatrix} m_1(b-b_1) \\ m_2(b-b_2) \\ \dots \\ m_n(b-b_n) \end{pmatrix}.$$

Our next application deals with *dynamic producer-consumer models.* Consider a market where a commodity or a service is supplied by n competing firms. Let x_k denote the output of firm k $(k = 1,2,\dots,n)$, and assume that $C_k(x_k) = B_k x_k^2 + b_k x_k + c_k$ is its cost function $(B_k, b_k, c_k > 0)$. At each time period, firm k maximizes its expected profit given its price prediction $p_k^E(t+1)$. The expected profit of this firm is the following

$$x_k p_k^E(t+1) - \left(B_k x_k^2 + b_k x_k + c_k\right),$$

and assuming interior optimum, simple differentiation shows that the profit maximizing output is given as

$$x_k(t+1) = \frac{1}{2B_k}\left(b_k - p_k^E(t+1)\right).$$

If firm k believes that the price does not change from time period t to $t+1$, then this firm selects $p_k^E(t+1) = p(t)$ and the profit maximizing output is given as

$$x_k(t+1) = \frac{1}{2B_k}(b_k - p(t)). \tag{1.35}$$

Let $d(t) = Dp(t) + d$ denote the market demand function, from which the price $p(t)$ can be obtained as

$$p(t) = \frac{1}{D}(d(t) - d). \tag{1.36}$$

If we assume that at each time period, the supply equals the demand, then $d(t) = \sum_{l=1}^{n} x_l(t)$, and substituting this equation and (1.36) into (1.35) the following difference equation is obtained:

$$\begin{aligned} x_k(t+1) &= \frac{1}{2B_k}\left(b_k - \frac{1}{D}\left(\sum_{l=1}^{n} x_l(t) - d\right)\right) \\ &= \sum_{l=1}^{n} \frac{1}{2B_k D} x_l(t) + \frac{1}{2B_k}\left(b_k + \frac{d}{D}\right), \end{aligned} \tag{1.37}$$

which is the special case of the difference equations (1.28) with

$$\underline{A} = \begin{pmatrix} \frac{1}{2B_1 D} & \frac{1}{2B_1 D} & \cdots & \frac{1}{2B_1 D} \\ \frac{1}{2B_2 D} & \frac{1}{2B_2 D} & \cdots & \frac{1}{2B_2 D} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2B_n D} & \frac{1}{2B_n D} & \cdots & \frac{1}{2B_n D} \end{pmatrix}, \quad \text{and} \quad \underline{b} = \begin{pmatrix} \frac{1}{2B_1}\left(b_1 + \frac{d}{D}\right) \\ \frac{1}{2B_2}\left(b_2 + \frac{d}{D}\right) \\ \cdots \\ \frac{1}{2B_n}\left(b_n + \frac{d}{D}\right) \end{pmatrix}.$$

The continuous counterpart of this model can be introduced as follows. Assume that at each time period $t \geq 0$, each firm adjusts its production output proportionally to its expected marginal profit:

$$\dot{x}_k(t) = m_k\left(p_k^E(t) - 2B_k x_k(t) - b_k\right),$$

where $p_k^E(t)$ is its price prediction for time period t. Assuming again that $p_k^E(t) = p(t)$, equation (1.36) holds, $d(t) = \sum_{l=1}^{n} x_l(t)$, and an ordinary differential equation is obtained for x_k:

$$\begin{aligned} \dot{x}_k(t) &= m_k\left(\frac{1}{D}\left(\sum_{l=1}^{n} x_l(t) - d\right) - 2B_k x_k(t) - b_k\right) \\ &= \left(\frac{m_k}{D} - 2B_k m_k\right) x_k(t) + \frac{m_k}{D}\sum_{l \neq k} x_l(t) + \left(-\frac{dm_k}{D} - m_k b_k\right) \end{aligned}$$

which is a special case of the differential equation (1.30) with

$$\underline{A} = \begin{pmatrix} m_1\left(\frac{1}{D}-2B_1\right) & \frac{m_1}{D} & \frac{m_1}{D} & \cdots & \frac{m_1}{D} \\ \frac{m_2}{D} & m_2\left(\frac{1}{D}-2B_2\right) & \frac{m_2}{D} & \cdots & \frac{m_2}{D} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m_n}{D} & \frac{m_n}{D} & \frac{m_n}{D} & \cdots & m_n\left(\frac{1}{D}-2B_n\right) \end{pmatrix}$$

and

$$\underline{b} = \begin{pmatrix} -m_1\left(\frac{d}{D}+b_1\right) \\ -m_2\left(\frac{d}{D}+b_2\right) \\ \cdots \\ -m_n\left(\frac{d}{D}+b_n\right) \end{pmatrix}. \tag{1.38}$$

Matrix multiplications can be illustrated by the simple example of a bakery. Assume that it makes three kinds of biscuits. The recipes are summarized in the following table:

	Flour	Sugar	Margarine
Type 1	0.5 (kg)	0.2 (kg)	0.3 (kg)
Type 2	0.55 (kg)	0.25 (kg)	0.2 (kg)
Type 3	0.6 (kg)	0.15 (kg)	0.25 (kg)

The quantities of the ingredients are given for one kilogram of each type. Let p_1, p_2, p_3 denote the prices per kg of the raw materials, then the material cost per kg of each biscuit type can be calculated as

$$c_1 = 0.5p_1 + 0.2p_2 + 0.3p_3$$
$$c_2 = 0.55p_1 + 0.25p_2 + 0.2p_3$$
$$c_3 = 0.6p_1 + 0.15p_2 + 0.25p_3.$$

By introducing the notation

$$\underline{A} = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.55 & 0.25 & 0.2 \\ 0.6 & 0.15 & 0.25 \end{pmatrix}, \ \underline{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

these relations can be rewritten as

$$\underline{c} = \underline{A}\underline{p}.$$

Assume next that the company sells three kinds of tins of assorted biscuits as given below:

	Type 1	Type 2	Type 3
Assortment 1	0.5 (kg)	0.7 (kg)	0.8 (kg)
Assortment 2	1.2 (kg)	0.3 (kg)	0.5 (kg)
Assortment 3	1 (kg)	0.5 (kg)	0.5 (kg)

The numbers show the amount of each type of biscuits in each type of assortment. Notice that each assortment contains 2 kg, and the costs of the tins are

$$k_1 = 0.5c_1 + 0.7c_2 + 0.8c_3$$
$$k_2 = 1.2c_1 + 0.3c_2 + 0.5c_3$$
$$k_3 = c_1 + 0.5c_2 + 0.5c_3.$$

Let

$$\underline{B} = \begin{pmatrix} 0.5 & 0.7 & 0.8 \\ 1.2 & 0.3 & 0.5 \\ 1 & 0.5 & 0.5 \end{pmatrix}, \text{ and } \underline{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix},$$

then we have

$$\underline{k} = \underline{B}\underline{c}.$$

Since $\underline{c} = \underline{A}\underline{p}$, this relation can be rewritten as

$$\underline{k} = \underline{B}\underline{A}\underline{p} = (\underline{B}\underline{A})\underline{p}$$

giving a direct relation between the prices of the ingredients and the costs of the assortments.

Matrices are used in describing directed graphs, which model, for example, material flows or many network problems. A directed graph is defined by a finite set of elements, $P_1, P_2, \ldots, P_n$, together with a finite collection of ordered pairs, (P_i, P_j), of distinct elements where no ordered pair is repeated. The elements $P_1, P_2, \ldots, P_n$ are called the vertices, and the ordered pairs are called the directed edges.

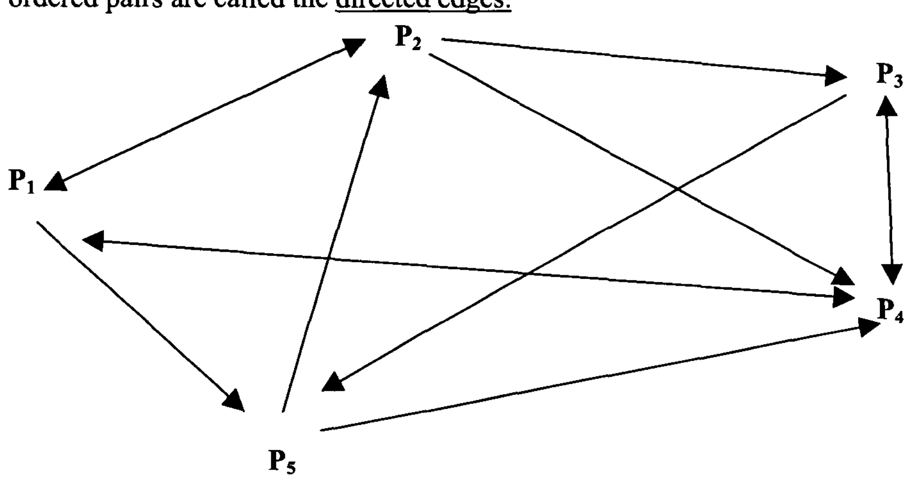

Figure 1.1 A directed graph with vertices

Figure 1.1 shows a directed graph with vertices $P_1,\ldots,P_5$, and edges (P_1, P_2), (P_1, P_4), (P_1, P_5), (P_2, P_1), (P_2, P_3), (P_2, P_4), (P_3, P_4), (P_3, P_5), (P_4, P_3), (P_4, P_1), (P_5, P_2), and (P_5, P_4). With a directed graph with n vertices, we may associate an $n\times n$ real matrix $\underline{A}$ with elements $a_{ij}= 1$ if (P_i, P_j) is a directed edge, otherwise $a_{ij}=0$. This matrix is called the vertex matrix of the directed graph. For example, in the previous example we select $n=5$, and

$$\underline{A}=\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

For any square matrix having 0 and 1 elements with all diagonal elements are zero, there exists a directed graph such that the given matrix is its vertex matrix. The elements of the vertex matrix show the direct connections from each vertex to the other vertices.

Let now $r\geq 1$ be a positive integer, and let $a_{ij}^{(r)}$ denote the (i, j)-element of the matrix $\underline{A}^r$. Then it is easy to show that $a_{ij}^{(r)}$ equals the number of r-step connections from P_i to P_j. In the case of the above matrix,

$$\underline{A}^2=\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 \\ 2 & 0 & 2 & 1 & 0 \end{pmatrix}.$$

The (1,1)-element with value 2 shows that there are two 2-step connections from vertex 1 to itself. From the graph we see that $P_1 \to P_2 \to P_1$ and $P_1 \to P_4 \to P_1$ are the two such connections.

The number of at most r-step connections from P_i to P_j is given as $a_{ij} + a_{ij}^{(2)} + \ldots + a_{ij}^{(r)}$. For example, the number of at most two-step connections are summarized in matrix

$$\underline{A} + \underline{A}^2 = \begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ 2 & 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 2 & 0 \end{pmatrix}.$$

Consider as an example, the (1,4) element of this matrix. The direct connection from P_1 to P_4 gives the only 1-step connection, and the two 2-step connections are $P_1 \to P_2 \to P_4$ and $P_1 \to P_5 \to P_4$.

1.6. Exercises

1. Specify a 3 * 4 matrix which has only positive entries.

2. Specify 5 different matrices which have entries equal to 1.

3. Compute

$$2 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix} - 3 \begin{pmatrix} 1 & 2 & 2 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Prove that, if $\underline{A}, \underline{B}, \underline{C}$ are matrices of the same size, then

$$(\underline{A}+\underline{B})-(\underline{A}+\underline{C})=\underline{B}-\underline{C}.$$

5. Solve equation

$$\underline{X}+\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix}=\begin{pmatrix}3 & 3 & 3\\ 4 & 4 & 4\end{pmatrix}.$$

What is the size of $\underline{X}$?

6. Solve equation

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix}+\underline{X}=\begin{pmatrix}3 & 3 & 3\\ 4 & 4 & 4\end{pmatrix}.$$

What is the size of $\underline{X}$? Explain why the results of this and the previous problems coincide.

7. Verify relations (1.11), (1.12), and (1.13).

8. Find values of a, b and c such that

$$\begin{pmatrix}1+a & b\\ b & c\end{pmatrix}+\begin{pmatrix}b & a\\ c & a\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

9. Let

$$\underline{A}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \underline{B}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}, \underline{C}=\begin{pmatrix}3 & 3\\ 3 & 3\end{pmatrix}.$$

Determine the results of the following operations:

a) $3\underline{A}+\underline{B}-2\underline{C}$

b) $4\underline{A}-\underline{B}+\underline{C}$

c) $-\underline{A}-\underline{B}+6\underline{C}$.

10. Solve equation

$$2(\underline{X}+\underline{A})+\frac{1}{2}(\underline{X}+\underline{B}-\underline{C})=\underline{X}$$

for $\underline{X}$, where $\underline{A},\underline{B}$, and $\underline{C}$ are the same as given in the previous problem.

11. Let

$$\underline{A}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \quad \text{and} \quad \underline{B}=\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}.$$

Compute the results of the following operations

a) $\underline{AB}$

b) $\underline{BA}$

c) $(\underline{AB})\underline{A}$

d) $((\underline{AB})\underline{A})\underline{B}$

e) $((\underline{AB})\underline{B})\underline{A}$

f) $\underline{B}(\underline{AB})\underline{A}$.

12. Show that $(\underline{A}+\underline{B})^2=\underline{A}^2+2\underline{AB}+\underline{B}^2$ if and only if $\underline{AB}=\underline{BA}$.

13. Show that $(\underline{A}+\underline{B})(\underline{A}-\underline{B})=\underline{A}^2-\underline{B}^2$ if and only if $\underline{AB}=\underline{BA}$.

14. Verify that matrix $\underline{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies equation $\underline{A}^2 = \underline{O}$.

15. Characterize all 2 * 2 real matrices such that $\underline{A}^2 = \underline{I}$.

16. Show that for all $k \geq 1$ and $n \times n$ matrices $\underline{A}$,

$$(\underline{I}_n - \underline{A})(\underline{I}_n + \underline{A} + \underline{A}^2 + \ldots + \underline{A}^k) = \underline{I}_n - \underline{A}^{k+1}.$$

17. Find a matrix satisfying equations

a) $\underline{X}^2 - 3\underline{X} + 2\underline{I} = \underline{O}$

b) $\underline{X}^3 - \underline{I} = \underline{O}$

c) $\underline{X}^3 - \underline{X}^2 - \underline{X} = \underline{O}$.

18. Assume that $ad - bc \neq 0$. Prove that
19.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

19. Find $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^{-1}$ by using the result of the previous problem.

20. Let $\underline{A} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$.

Find matrices $\underline{B}$, $\underline{C}$ such that $\underline{B} > \underline{A}$ and $\underline{C} < \underline{A}$.

21. Let $\underline{A} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ and $\underline{B} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$. Find a matrix $\underline{C}$ such that $\underline{C} > \underline{A} - \underline{B}$.

22. Is $(\underline{A} + \underline{B})^{-1} = \underline{A}^{-1} + \underline{B}^{-1}$?

23. Let $\underline{p}^T = (p_1, p_2, \ldots, p_n)$ be a price vector, and let $\underline{1}$ be the n-element vector all elements of which are equal to 1. Show that the average price can be expressed as $\frac{1}{n}\underline{p}^T\underline{1}$.

24. Find $\underline{A}^t$ for matrix $\underline{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

25. Find $e^{\underline{A}t}$ for matrix $\underline{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.