

3. Moving Flux Lines and Polymers

3.1. Some properties of the KPZ equation

First consider *deterministic growth*, such as a slow and uniform snowfall, on an initial profile which at $t = 0$ is described by $h_0(\mathbf{x})$. The nonlinear equation can in fact be *linearized* with the aid of a ‘‘Cole-Hopf’’ transformation,

$$W(\mathbf{x}, t) = \exp \left[\frac{\lambda}{2\nu} h(\mathbf{x}, t) \right]. \quad (82)$$

The function $W(\mathbf{x}, t)$ evolves according to the diffusion equation with *multiplicative noise*,

$$\frac{\partial W(\mathbf{x}, t)}{\partial t} = \nu \nabla^2 W + \frac{\lambda}{2\nu} W \eta(\mathbf{x}, t). \quad (83)$$

In the absence of noise, $\eta(\mathbf{x}, t) = 0$, Eq. (83) can be solved subject to the initial condition $W(\mathbf{x}, t = 0) = \exp[\lambda h_0(\mathbf{x})/2\nu]$, and leads to the growth profile,

$$h(\mathbf{x}, t) = \frac{2\nu}{\lambda} \ln \left\{ \int d^d \mathbf{x}' \exp \left[-\frac{|\mathbf{x} - \mathbf{x}'|^2}{2\nu t} + \frac{\lambda}{2\nu} h(\mathbf{x}', t) \right] \right\}. \quad (84)$$

It is instructive to examine the $\nu \rightarrow 0$ limit, which is indeed appropriate to snowfalls since there is not much rearrangement after deposition. In this limit, the integral in Eq. (84) can be performed by the saddle point method. For each \mathbf{x} we have to identify a point \mathbf{x}' which maximizes the exponent, leading to a collection of paraboloids described by

$$h(\mathbf{x}, t) = \max_{\mathbf{x}'} \left\{ h_0(\mathbf{x}') - \frac{|\mathbf{x} - \mathbf{x}'|^2}{2\lambda t} \right\}. \quad (85)$$

Such parabolic sequences are quite common in many layer by layer growth processes in nature, from biological to geological formations. The patterns for $\lambda = 1$ are identical to those obtained by the geometrical construction of Huygens, familiar from optics. The growth profile (wave front) is constructed from the outer envelop of circles of radius t drawn from all points on the initial profile. The nonlinearity in Eq. (71) thus algebraically accounts for their origin.

As growth proceeds, the surface smoothens by the *coarsening* of the paraboloids. What is the typical size of the features at time t ? In maximizing the exponent in Eq. (85), we have to balance a reduction $|\mathbf{x} - \mathbf{x}'|^2/2\lambda t$, by a possible

gain from $h_0(\mathbf{x}')$ in selecting a point away from \mathbf{x} . The final scaling is controlled by the roughness of the initial profile. Let us assume that the original pattern is a *self-affine fractal* of roughness χ , i.e.

$$|h_0(\mathbf{x}) - h_0(\mathbf{x}')| \sim |\mathbf{x} - \mathbf{x}'|^\chi. \quad (86)$$

Balancing the two terms in Eq. (85) gives

$$(\delta x)^\chi \sim \frac{(\delta x)^2}{t} \implies \delta x \sim t^{1/z}, \quad \text{with } z + \chi = 2. \quad (87)$$

For example, if the initial profile is like a random walk in $d = 1$, $\chi = 1/2$, and $z = 3/2$. This leads to the spreading of information along the profile by a process that is faster than diffusion, $\delta x \sim t^{2/3}$.

Note that the slope, $\vec{v}(\mathbf{x}, t) = -\lambda \vec{\nabla} h(x, t)$, satisfies the equation,

$$\frac{D\vec{v}(\mathbf{x}, t)}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = \nu \nabla^2 \vec{v} - \lambda \nabla \eta. \quad (88)$$

This is the Navier–Stokes equation for the velocity of a fluid of viscosity ν , which is being randomly stirred by a conservative force,³⁴ $\vec{f} = -\lambda \nabla \eta$. The fluid is vorticity free since

$$\vec{\Omega} = \vec{\nabla} \times \vec{v} = -\lambda \nabla \times \nabla h = 0. \quad (89)$$

This is the *Burgers'* equation,³⁵ which provides a simple example of the formation of shock waves in a fluid. The gradient of Eq. (85) in $d = 1$ gives a saw-tooth pattern of shocks which coarsen in time.

To study stochastic roughening in the presence of the nonlinear term, we first carry out a scaling analysis. Under the scaling $\mathbf{x} \rightarrow b\mathbf{x}$, $t \rightarrow b^z t$, and $h \rightarrow b^x h$, Eq. (71) transforms to

$$b^{x-z} \frac{\partial h}{\partial t} = \nu b^{x-2} \nabla^2 h + \frac{\lambda}{2} b^{2x-2} (\nabla h)^2 + \eta(b\mathbf{x}, b^z t). \quad (90)$$

The correlations of the transformed noise, $\eta'(\mathbf{x}, t) = b^{z-x} \eta(b\mathbf{x}, b^z t)$, satisfy

$$\begin{aligned} \langle \eta'(\mathbf{x}, t) \eta'(\mathbf{x}', t') \rangle &= b^{2z-2x} 2D \delta^d(\mathbf{x} - \mathbf{x}') b^{-d} \delta(t - t') b^{-z} \\ &= b^{z-d-2x} 2D \delta^d(\mathbf{x} - \mathbf{x}') \delta(t - t'). \end{aligned} \quad (91)$$

Under such scaling the parameters of Eq. (71) are transformed to

$$\begin{cases} \nu \rightarrow b^{z-2} \nu \\ \lambda \rightarrow b^{x+z-2} \lambda \\ D \rightarrow b^{z-2x-d} D. \end{cases} \quad (92)$$

For $\lambda = 0$, the equation is made scale invariant upon the choice of $z_0 = 2$, and $\chi_0 = (2 - d)/2$. Close to this linear-fixed point, λ scales to $b^{z_0 + \chi_0 - 2}\lambda = b^{(2-d)/2}\lambda$, and is a relevant operator for $d < 2$. In fact a perturbative dynamic renormalization group suggests that it is *marginally relevant* at $d = 2$, and that in all dimensions a sufficiently large λ leads to new scaling behavior.

Are there any nonrenormalization conditions that help in identifying the exponents of the full nonlinear stochastic equation? Note that since Eqs. (71) and (88) are related by a simple transformation, they must have the same scaling properties. Since the Navier–Stokes equation is derivable from Newton’s laws of motion for fluid particles, it has the Galilean invariance of changing to a uniformly moving coordinate frame. This symmetry is preserved under renormalization to larger scales and requires that the ratio of the two terms on the left-hand side of Eq. (88) ($\partial_t \vec{v}$ and $\vec{v} \cdot \nabla \vec{v}$) stays at unity. In terms of Eq. (71), this implies the nonrenormalization of the parameter λ , and leads to the exponent identity

$$\chi + z = 2. \quad (93)$$

Unfortunately there is no other nonrenormalization condition except in $d = 1$. Following Eq. (48), we can write down a Fokker–Planck equation for the evolution of the configurational probability as

$$\frac{\partial \mathcal{P}([h(\mathbf{x})], t)}{\partial t} = - \int d^d \mathbf{x} \frac{\delta}{\delta h(\mathbf{x})} \left[\left(\nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \right) \mathcal{P} - D \frac{\delta \mathcal{P}}{\delta h(\mathbf{x})} \right]. \quad (94)$$

Since Eq. (71) was not constructed from a Hamiltonian in general, we do not know the stationary solution at long times. In $d = 1$, we make a guess and try a solution of the form

$$\mathcal{P}_0[h(x)] \propto \exp \left[-\frac{\nu}{2D} \int dx (\partial_x h)^2 \right]. \quad (95)$$

Since

$$\frac{\delta \mathcal{P}_0}{\delta h(x)} = -\partial_x \frac{\delta \mathcal{P}_0}{\delta (\partial_x h)} = \frac{\nu}{D} (\partial_x^2 h) \mathcal{P}_0, \quad (96)$$

Equation (94) leads to

$$\begin{aligned} \frac{\partial \mathcal{P}_0}{\partial t} &= - \int dx \frac{\delta \mathcal{P}_0}{\delta h(x)} \left(\nu \partial_x^2 h + \frac{\lambda}{2} (\partial_x h)^2 - D \frac{\nu}{D} \partial_x^2 h \right) \\ &= -\frac{\lambda}{2} \mathcal{P}_0 \int dx \frac{\nu}{D} (\partial_x^2 h) (\partial_x h)^2 = -\frac{\lambda \nu}{2D} \mathcal{P}_0 \int dx \partial_x \left(\frac{(\partial_x h)^3}{3} \right) = 0. \end{aligned} \quad (97)$$

We have thus identified the stationary state of the one-dimensional equation. (This procedure does not work in higher dimensions as it is impossible to write the final result as a total derivative.) Surprisingly, the stationary distribution is the same as the one in equilibrium at a temperature proportional to D/ν . We can thus immediately identify the roughness exponent $\chi = 1/2$, which together with the exponent identity in Eq. (93) leads to $z = 3/2$, i.e. superdiffusive behavior.

The values of the exponents in the strongly nonlinear regime are not known exactly in higher dimensions. However, extensive numerical simulations of growth have provided fairly reliable estimates.¹¹ In the physically relevant case ($d = 2$) of a surface grown in three dimensions, $\chi \approx 0.39$ and $z \approx 1.61$.³⁶

As an aside, we remark that some exact information is available for the *anisotropic* KPZ equation in $2 + 1$ dimensions. Using a perturbative RG approach, Wolf showed³⁷ that in the equation

$$\partial_t h = K \nabla^2 h + \frac{\lambda_x}{2} (\partial_x h)^2 + \frac{\lambda_y}{2} (\partial_y h)^2 + \eta(x, y, t), \quad (98)$$

the nonlinearities $\{\lambda_x, \lambda_y\}$ renormalize to zero if they initially have opposite signs. This suggests logarithmic fluctuations for the resulting interface, as in the case of the linear Langevin equation. In fact, it is straightforward to demonstrate that Eq. (98) also satisfies a fluctuation dissipation condition if $\lambda_x = -\lambda_y$. When this condition is satisfied, the associated Fokker-Planck equation has a steady-state solution

$$\mathcal{P} = \exp \left(-\frac{\nu}{2D} \int dx dy (\nabla h)^2 \right). \quad (99)$$

This is a nonperturbative result which again indicates the logarithmic fluctuations resulting from Eq. (98). In this context, it is interesting to note that the steady-state distribution for an exactly solvable discrete model of surface growth belonging to the above universality class has also been obtained in Ref. 38. (We have shown that there are no other similarly generalized KPZ equations that satisfy a fluctuation dissipation condition in higher dimensions.³⁹)

3.2. A moving flux line

Let us now turn to the case of a line in three dimensions. Fluctuations of the line can be indicated by a two dimensional vector \mathbf{r} . (The notations for

the parameters in this section are chosen to conform with the literature for polymers.) Even in an isotropic medium, the drift velocity \mathbf{v} breaks the isotropy in \mathbf{r} by selecting a direction. A gradient expansion up to second order for the equation of motion gives⁴⁰

$$\partial_t r_\alpha = [K_1 \delta_{\alpha\beta} + K_2 v_\alpha v_\beta] \partial_x^2 r_\beta + [\lambda_1 (\delta_{\alpha\beta} v_\gamma + \delta_{\alpha\gamma} v_\beta) + \lambda_2 v_\alpha \delta_{\beta\gamma} + \lambda_3 v_\alpha v_\beta v_\gamma] \frac{\partial_x r_\beta \partial_x r_\gamma}{2} + \eta_\alpha, \quad (100)$$

with random force correlations

$$\langle \eta_\alpha(x, t) \eta_\beta(x', t') \rangle = 2[T_1 \delta_{\alpha\beta} + T_2 v_\alpha v_\beta] \delta(x - x') \delta(t - t'). \quad (101)$$

Higher order nonlinearities can be similarly constructed but are in fact irrelevant. In terms of components parallel and perpendicular to the velocity, the equations are

$$\begin{cases} \partial_t r_\parallel = K_\parallel \partial_x^2 r_\parallel + \frac{\lambda_\parallel}{2} (\partial_x r_\parallel)^2 + \frac{\lambda_\times}{2} (\partial_x r_\perp)^2 + \eta_\parallel(x, t), \\ \partial_t r_\perp = K_\perp \partial_x^2 r_\perp + \lambda_\perp \partial_x r_\parallel \partial_x r_\perp + \eta_\perp(x, t), \end{cases} \quad (102)$$

with

$$\begin{cases} \langle \eta_\parallel(x, t) \eta_\parallel(x', t') \rangle = 2T_\parallel \delta(x - x') \delta(t - t'), \\ \langle \eta_\perp(x, t) \eta_\perp(x', t') \rangle = 2T_\perp \delta(x - x') \delta(t - t'). \end{cases} \quad (103)$$

The noise-averaged correlations have the dynamic scaling form

$$\begin{cases} \langle [r_\parallel(x, t) - r_\parallel(x', t')]^2 \rangle = |x - x'|^{2\zeta_\parallel} g_\parallel \left(\frac{|t - t'|}{|x - x'|^{z_\parallel}} \right), \\ \langle [r_\perp(x, t) - r_\perp(x', t')]^2 \rangle = |x - x'|^{2\zeta_\perp} g_\perp \left(\frac{|t - t'|}{|x - x'|^{z_\perp}} \right). \end{cases} \quad (104)$$

In the absence of nonlinearities ($\lambda_\parallel = \lambda_\times = \lambda_\perp = 0$), Eqs. (102) can easily be solved to give $\zeta_\parallel = \zeta_\perp = 1/2$ and $z_\parallel = z_\perp = 2$. Simple dimensional counting indicates that all three nonlinear terms are relevant and may modify the exponents in Eq. (104). Studies of related stochastic equations^{41,37} indicate that interesting dynamic phase diagrams may emerge from the competition between nonlinearities. Let us assume that λ_\parallel is positive and finite (its sign

can be changed by $r_{\parallel} \rightarrow -r_{\parallel}$), and focus on the dependence of the scaling exponents on the ratios $\lambda_{\perp}/\lambda_{\parallel}$ and $\lambda_{\times}/\lambda_{\parallel}$. (It is more convenient to set the vertical axis to $\lambda_{\times}K_{\parallel}T_{\perp}/\lambda_{\parallel}K_{\perp}T_{\parallel}$.)

The properties discussed for the KPZ equation can be extended to this higher dimensional case:

- (1) *Galilean Invariance (GI)*: Consider the infinitesimal reparametrization

$$\begin{cases} x' = x + \lambda_{\parallel}\epsilon t, & t' = t, \\ r'_{\parallel} = r_{\parallel} + \epsilon x, & r'_{\perp} = r_{\perp}. \end{cases} \quad (105)$$

Equations (102) are invariant under this transformation provided that $\lambda_{\parallel} = \lambda_{\perp}$. Thus *along this line* there is GI, which again implies the exponent identity

$$\zeta_{\parallel} + z_{\parallel} = 2. \quad (106)$$

- (2) *Fluctuation–Dissipation (FD) Condition*: The Fokker–Planck equation for the evolution of the joint probability $\mathcal{P}[r_{\parallel}(x), r_{\perp}(x)]$ has a stationary solution

$$\mathcal{P}_0 \propto \exp\left(-\int dx \left[\frac{K_{\parallel}}{2T_{\parallel}}(\partial_x r_{\parallel})^2 + \frac{K_{\perp}}{2T_{\perp}}(\partial_x r_{\perp})^2\right]\right), \quad (107)$$

provided that $\lambda_{\times}K_{\parallel}T_{\perp} = \lambda_{\perp}K_{\perp}T_{\parallel}$. Thus for this special choice of parameters, if \mathcal{P} converges to this solution, the long-time behavior of the correlation functions in Eq. (104) can be directly read off Eq. (107), giving $\zeta_{\parallel} = \zeta_{\perp} = 1/2$.

- (3) *The Cole–Hopf (CH) Transformation*: is an important method for the exact study of solutions of the KPZ equation. Here we generalize this transformation to the complex plane by defining, for $\lambda_{\times} < 0$,

$$\Psi(x, t) = \exp\left(\frac{\lambda_{\parallel}r_{\parallel}(x, t) + i\sqrt{-\lambda_{\parallel}\lambda_{\times}}r_{\perp}(x, t)}{2K}\right). \quad (108)$$

The linear diffusion equation

$$\partial_t \Psi = K \partial_x^2 \Psi + \mu(x, t) \Psi,$$

then leads to Eqs. (102) if $K_{\parallel} = K_{\perp} = K$ and $\lambda_{\parallel} = \lambda_{\perp}$. [Here $\text{Re}(\mu) = \lambda_{\parallel}\eta_{\parallel}/2K$ and $\text{Im}(\mu) = \sqrt{-\lambda_{\parallel}\lambda_{\times}}\eta_{\perp}/2K$.] This transformation enables an

exact solution of the *deterministic* equation, and further allows us to write the solution to the *stochastic* equation in the form of a path integral

$$\Psi(x, t) = \int_{(0,0)}^{(x,t)} \mathcal{D}x(\tau) \exp \left\{ - \int_0^t d\tau \left[\frac{\dot{x}^2}{2K} + \mu(x, \tau) \right] \right\}. \quad (109)$$

Equation (109) has been extensively studied in connection with quantum tunneling in a disordered medium,⁴² with Ψ representing the wave function. In particular, results for the tunneling probability $|\Psi|^2$ suggest $z_{\parallel} = 3/2$ and $\zeta_{\parallel} = 1/2$. The transverse fluctuations correspond to the phase in the quantum problem which is not observable. Hence this mapping does not provide any information on ζ_{\perp} and z_{\perp} , which are in fact observable for the moving line.

At the point $\lambda_{\perp} = \lambda_{\times} = 0$, r_{\parallel} and r_{\perp} decouple, and $z_{\perp} = 2$ while $z_{\parallel} = 3/2$. However, in general $z_{\parallel} = z_{\perp} = z$ unless the effective λ_{\perp} is zero. For example at the intersection of the subspaces, with GI and FD, the exponents $z_{\parallel} = z_{\perp} = 3/2$ are obtained from the exponent identities. Dynamic RG recursion relations can be computed to one-loop order,^{40,43} by standard methods of momentum-shell dynamic RG.^{34,31}

The renormalization of the seven parameters in Eqs. (102), generalized to n transverse directions, give the recursion relations

$$\begin{aligned} \frac{dK_{\parallel}}{d\ell} &= K_{\parallel} \left[z - 2 + \frac{1}{\pi} \frac{\lambda_{\parallel}^2 T_{\parallel}}{4K_{\parallel}^3} + n \frac{1}{\pi} \frac{\lambda_{\perp} \lambda_{\times} T_{\perp}}{4K_{\parallel} K_{\perp}^2} \right], \\ \frac{dK_{\perp}}{d\ell} &= K_{\perp} \left[z - 2 + \frac{1}{\pi} \frac{\lambda_{\perp} ((\lambda_{\times} T_{\perp}/K_{\perp}) + (\lambda_{\perp} T_{\parallel}/K_{\parallel}))}{2K_{\perp} (K_{\perp} + K_{\parallel})} \right. \\ &\quad \left. + \frac{1}{\pi} \frac{K_{\perp} - K_{\parallel}}{K_{\perp} + K_{\parallel}} \frac{\lambda_{\perp} ((\lambda_{\times} T_{\perp}/K_{\perp}) - (\lambda_{\perp} T_{\parallel}/K_{\parallel}))}{K_{\perp} (K_{\perp} + K_{\parallel})} \right], \\ \frac{dT_{\parallel}}{d\ell} &= T_{\parallel} \left[z - 2\zeta_{\parallel} - 1 + \frac{1}{\pi} \frac{\lambda_{\parallel}^2 T_{\parallel}}{4K_{\parallel}^3} \right] + n \frac{1}{\pi} \frac{\lambda_{\times}^2 T_{\perp}^2}{4K_{\perp}^3}, \\ \frac{dT_{\perp}}{d\ell} &= T_{\perp} \left[z - 2\zeta_{\perp} - 1 + \frac{1}{\pi} \frac{\lambda_{\perp}^2 T_{\parallel}}{K_{\perp} K_{\parallel} (K_{\perp} + K_{\parallel})} \right], \\ \frac{d\lambda_{\parallel}}{d\ell} &= \lambda_{\parallel} [\zeta_{\parallel} + z - 2], \end{aligned} \quad (110)$$

$$\frac{d\lambda_{\perp}}{d\ell} = \lambda_{\perp} \left[\zeta_{\parallel} + z - 2 - \frac{1}{\pi} \frac{\lambda_{\parallel} - \lambda_{\perp}}{(K_{\perp} + K_{\parallel})^2} ((\lambda_{\times} T_{\perp}/K_{\perp}) - (\lambda_{\perp} T_{\parallel}/K_{\parallel})) \right],$$

$$\frac{d\lambda_{\times}}{d\ell} = \lambda_{\times} \left[2\zeta_{\perp} - \zeta_{\parallel} + z - 2 + \frac{1}{\pi} \frac{\lambda_{\parallel} K_{\perp} - \lambda_{\perp} K_{\parallel}}{K_{\perp} K_{\parallel} (K_{\perp} + K_{\parallel})} ((\lambda_{\times} T_{\perp}/K_{\perp}) - (\lambda_{\perp} T_{\parallel}/K_{\parallel})) \right].$$

The RG flows naturally satisfy the constraints imposed by the nonperturbative results: the subspace of GI is closed under RG, while the FD condition appears as a *fixed line*. The RG flows, and the corresponding exponents, are different in each quadrant of parameter space, which implies that the scaling behavior is determined by the relative signs of the three nonlinearities. This was confirmed by numerical integrations^{40,43} of Eqs. (102), performed for different sets of parameters.

The analysis of analytical and numerical results can be summarized as follows:

- $\lambda_{\perp} \lambda_{\times} > 0$: In this region, the scaling behavior is understood best. The RG flows terminate on the fixed line where FD conditions apply, hence $\zeta_{\parallel} = \zeta_{\perp} = 1/2$. All along this line, the one loop RG exponent is $z = 3/2$. These results are consistent with the numerical simulations. The measured exponents rapidly converge to these values, except when λ_{\perp} or λ_{\times} are small.
- $\lambda_{\times} = 0$: In this case the equation for r_{\parallel} is the KPZ equation (71), thus $\zeta_{\parallel} = 1/2$ and $z_{\parallel} = 3/2$. The fluctuations in r_{\parallel} act as a strong (multiplicative and correlated) noise on r_{\perp} . The one-loop RG yields the exponents $z_{\perp} = 3/2$, $\zeta_{\perp} = 0.75$ for $\lambda_{\perp} > 0$, while a negative λ_{\perp} scales to 0 suggesting $z_{\perp} > z_{\parallel}$. Simulations are consistent with the RG calculations for $\lambda_{\perp} > 0$, yielding $\zeta_{\perp} = 0.72$, surprisingly close to the one-loop RG value. For $\lambda_{\perp} < 0$, simulations indicate $z_{\perp} \approx 2$ and $\zeta_{\perp} \approx 2/3$ along with the expected values for the longitudinal exponents.
- $\lambda_{\perp} = 0$: The transverse fluctuations satisfy a simple diffusion equation with $\zeta_{\perp} = 1/2$ and $z_{\perp} = 2$. Through the term $\lambda_{\times} (\partial_x r_{\perp})^2/2$, these fluctuations act as a correlated noise³¹ for the longitudinal mode. A naive application of the results of this Ref. 31 gives $\zeta_{\parallel} = 2/3$ and $z_{\parallel} = 4/3$. Quite surprisingly, simulations indicate different behavior depending on the sign of λ_{\times} . For $\lambda_{\times} < 0$, $z_{\parallel} \approx 3/2$ and $\zeta_{\parallel} \approx 1/2$ whereas for $\lambda_{\times} > 0$, longitudinal fluctuations are much stronger, resulting in $z_{\parallel} \approx 1.18$ and $\zeta_{\parallel} \approx 0.84$. Actually, ζ_{\parallel} increases

steadily with system size, suggesting a breakdown of dynamic scaling, due to a change of sign in $\lambda_{\perp}\lambda_{\times}$.

- $\lambda_{\perp} < 0$ and $\lambda_{\times} > 0$: The analysis of this region is the most difficult in that the RG flows do not converge upon a finite fixed point and $\lambda_{\perp} \rightarrow 0$, which may signal the breakdown of dynamic scaling. Simulations indicate strong longitudinal fluctuations that lead to instabilities in the discrete integration scheme, excluding the possibility of measuring the exponents reliably.
- $\lambda_{\perp} > 0$ and $\lambda_{\times} < 0$: The *projected* RG flows in this quadrant converge to the point $\lambda_{\perp}/\lambda_{\parallel} = 1$ and $\lambda_{\times}T_{\perp}K_{\parallel}/\lambda_{\parallel}T_{\parallel}K_{\perp} = -1$. This is actually not a fixed point, as K_{\parallel} and K_{\perp} scale to infinity. The applicability of the CH transformation to this point implies $z_{\parallel} = 3/2$ and $\zeta_{\parallel} = 1/2$. Since λ_{\perp} is finite, $z_{\perp} = z_{\parallel} = 3/2$ is expected, but this does not give any information on ζ_{\perp} . Simulations indicate strong transverse fluctuations and suffer from difficulties similar to those in the previous region.

Equations (102) are the simplest nonlinear, local, and dissipative equations that govern the fluctuations of a moving line in a random medium. They can be easily generalized to describe the time evolution of a manifold with arbitrary internal ($\mathbf{x} \in R^d$) and external ($\mathbf{r} \in R^{n+1}$) dimensions, and to the motion of curves that are not necessarily stretched in a particular direction. Since the derivation only involves general symmetry arguments, the given results are widely applicable to a number of seemingly unrelated systems. We will discuss one application to drifting polymers in more detail in the next section, explicitly demonstrating the origin of the nonlinear terms starting from more fundamental hydrodynamic equations. A simple model of crack front propagation in three dimensions⁴⁴ also arrives at Eqs. (102), implying the self-affine structure of the crack surface after the front has passed.

3.3. Drifting polymers

The dynamics of polymers in fluids is of much theoretical interest and has been extensively studied.^{45,46} The combination of polymer flexibility, interactions, and hydrodynamics make a first principles approach to the problem quite difficult. There are, however, a number of phenomenological studies that describe various aspects of this problem.⁴⁷ One of the simplest is the Rouse model⁴⁸: The configuration of the polymer at time t is described by a vector $\mathbf{R}(x, t)$, where $x \in [0, N]$ is a continuous variable replacing the discrete monomer index.

Ignoring inertial effects, the relaxation of the polymer in a viscous medium is approximated by

$$\partial_t \mathbf{R}(x, t) = \mu \mathbf{F}(\mathbf{R}(x, t)) = K \partial_x^2 \mathbf{R}(x, t) + \eta(x, t), \quad (111)$$

where μ is the mobility. The force \mathbf{F} has a contribution from interactions with near neighbors that are treated as springs. Steric and other interactions are ignored. The effect of the medium is represented by the random velocities η with zero mean. The Rouse model is a linear Langevin equation that is easily solved. It predicts that the mean square radius of gyration, $R_g^2 = \langle |\mathbf{R} - \langle \mathbf{R} \rangle|^2 \rangle$, is proportional to the polymer size N , and the largest relaxation times scale as the fourth power of the wave number (i.e. in dynamic light scattering experiments, the half width at half maximum of the scattering amplitude scales as the fourth power of the scattering wave vector \mathbf{q}). These results can be summarized as $R_g \sim N^\nu$ and $\Gamma(\mathbf{q}) \sim q^z$, where ν and z are called the *swelling* and *dynamic* exponents, respectively.⁴⁹ Thus, for the Rouse Model, $\nu = 1/2$ and $z = 4$.

The Rouse model ignores hydrodynamic interactions mediated by the fluid. These effects were originally considered by Kirkwood and Risemann⁵⁰ and later on by Zimm.⁵¹ The basic idea is that the motion of each monomer modifies the flow field at large distances. Consequently, each monomer experiences an additional velocity

$$\begin{aligned} \delta_H \partial_t \mathbf{R}(x, t) &= \frac{1}{8\pi\eta_s} \int dx' \frac{\mathbf{F}(x') r_{xx'}^2 + (\mathbf{F}(x') \cdot \mathbf{r}_{xx'}) \mathbf{r}_{xx'}}{|\mathbf{r}_{xx'}|^3} \\ &\approx \int dx' \frac{\gamma}{|x - x'|^\nu} \partial_x^2 \mathbf{R}, \end{aligned} \quad (112)$$

where $\mathbf{r}_{xx'} = \mathbf{R}(x) - \mathbf{R}(x')$ and the final approximation is obtained by replacing the actual distance between two monomers by their average value. The modified equation is still linear in \mathbf{R} and easily solved. The main result is the speeding up of the relaxation dynamics as the exponent z changes from 4 to 3. Most experiments on polymer dynamics⁵² indeed measure exponents close to 3. Rouse dynamics is still important in other circumstances, such as diffusion of a polymer in a solid matrix, stress and viscoelasticity in concentrated polymer solutions, and is also applicable to relaxation times in Monte Carlo simulations.

Since both of these models are linear, the dynamics remains invariant in the center of mass coordinates upon the application of a uniform external force.

Hence the results for a drifting polymer are identical to a stationary one. This conclusion is in fact not correct due to the hydrodynamic interactions. For example, consider a rodlike conformation of the polymer with monomer length b_0 , where $\partial_x R_\alpha = b_0 t_\alpha$ everywhere on the polymer, so that the elastic (Rouse) force vanishes. If a uniform force \mathbf{E} per monomer acts on this rod, the velocity of the rod can be solved using Kirkwood Theory, and the result is⁴⁵

$$\mathbf{v} = \frac{(-\ln \kappa)}{4\pi\eta_s b_0} \mathbf{E} \cdot [\mathbf{I} + \mathbf{t}\mathbf{t}]. \quad (113)$$

In the above equation, η_s is the solvent viscosity, \mathbf{t} is the unit tangent vector, and $\kappa = 2b/b_0N$ is the ratio of the width b to the half length $b_0N/2$ of the polymer. A more detailed calculation of the velocity in the more general case of an arbitrarily shaped slender body by Khayat and Cox⁵³ shows that *nonlocal* contributions to the hydrodynamic force, which depend on the whole shape of the polymer rather than the local orientation, are $\mathcal{O}(1/(\ln \kappa)^2)$. Therefore, corrections to Eq. (113) are small when $N \gg b/b_0$.

Incorporating this tilt dependence of polymer mobility requires adding terms nonlinear in the tilt, $\partial_x \mathbf{r}$, to a *local* equation of motion. Since the overall force (or velocity) is the only vector breaking the isotropy of the fluid, the structure of these nonlinear terms must be identical to Eq. (100). Thus in terms of the fluctuations parallel and perpendicular to the average drift, we again recover the equations,

$$\begin{cases} \partial_t R_{\parallel} = U_{\parallel} + K_{\parallel} \partial_x^2 R_{\parallel} + \frac{\lambda_{\parallel}}{2} (\partial_x R_{\parallel})^2 + \frac{\lambda_{\times}}{2} \sum_{i=1}^2 (\partial_x R_{\perp i})^2 + \eta_{\parallel}(x, t), \\ \partial_t R_{\perp i} = K_{\perp} \partial_x^2 R_{\perp i} + \lambda_{\perp} \partial_x R_{\parallel} \partial_x R_{\perp i} + \eta_{\perp i}(x, t), \end{cases} \quad (114)$$

where $\{\perp i\}$ refers to the two transverse coordinates of the monomer positions. The noise is assumed to be white and gaussian but need not be isotropic, i.e.

$$\begin{cases} \langle \eta_{\parallel}(x, t) \eta_{\parallel}(x', t') \rangle = 2T_{\parallel} \delta(x - x') \delta(t - t'), \\ \langle \eta_{\perp i}(x, t) \eta_{\perp j}(x', t') \rangle = 2T_{\perp} \delta_{i,j} \delta(x - x') \delta(t - t'). \end{cases} \quad (115)$$

At zero average velocity, the system becomes isotropic and the equations of motion must coincide with the Rouse model. Therefore, $\{\lambda_{\parallel}, \lambda_{\times}, \lambda_{\perp}, U, K_{\parallel} - K_{\perp}, T_{\parallel} - T_{\perp}\}$ are all proportional to E for small forces. The relevance of these nonlinear terms are determined by the dimensionless scaling variable

$$y = \left(\frac{U}{U^*} \right) N^{1/2},$$

where U^* is a characteristic microscopic velocity associated with monomer motion and is roughly 10–20 m/s for polystyrene in benzene. The variable y is proportional to another dimensionless parameter, the Reynolds number Re , which determines the breakdown of hydrodynamic equations and onset of turbulence. However, typically $Re \ll y$, and the hydrodynamic equations are valid for moderately large y . Equations (114) describe the static and dynamical scaling properties of the nonlinear and anisotropic regime when $U > U^* N^{-1/2}$.

Equations (114) is just a slight variation from Eqs. (102), with two transverse components instead of one. Thus, the results discussed in the previous lecture apply. A more detailed calculation of the nonlinear terms from hydrodynamics⁵⁴ shows that all three nonlinearities are positive for small driving forces. In this case, the asymptotic scaling exponents are isotropic, with $\nu = 1/2$ and $z = 3$. However, the fixed points of the RG transformation are in general anisotropic, which implies a kinetically induced form birefringence *in the absence of external velocity gradients*. This is in marked contrast with standard theories of polymer dynamics where a uniform driving force has essentially no effect on the internal modes of the polymer.

When one of the nonlinearities approaches zero, the swelling exponents may become anisotropic and the polymer elongates or compresses along the longitudinal direction. However, the experimental path in the parameter space as a function of E is not known and not all of the different scaling regimes correspond to actual physical situations. The scaling results found by the RG analysis are verified by direct integration of equations, as mentioned in the earlier sections. A more detailed discussion of the analysis and results can be found in our earlier work.⁴³

In constructing Eqs. (114), we only allowed for local effects, and ignored the nonlocalities that are the hallmark of hydrodynamics. One consequence of hydrodynamic interactions is the *back-flow* velocity in Eq. (112) that can be added to the evolution Eqs. (114). Dimensional analysis gives the recursion relation

$$\frac{\partial \gamma}{\partial \ell} = \gamma[\nu z - 1 - (d - 2)\nu] + O(\gamma^2), \quad (116)$$

which implies that, at the nonlinear fixed point, this additional term is surprisingly irrelevant for $d > 3$, and $z = 3$ due to the nonlinearities. For $d < 3$, $z = d$ due to hydrodynamics, and the nonlinear terms are irrelevant. The situation in three dimensions is unclear, but a change in the exponents is unlikely. Similarly, one could consider the effect of self-avoidance by including the force

generated by a softly repulsive contact potential

$$\frac{b}{2} \int dx dx' \mathcal{V}(\mathbf{r}(x) - \mathbf{r}(x')). \quad (117)$$

The relevance of this term is also controlled by the scaling dimension $y_b = \nu z - 1 - (d - 2)\nu$, and therefore this effect is marginal in three dimensions at the nonlinear fixed point, in contrast with both Rouse and Zimm models where self-avoidance becomes relevant below four dimensions. Unfortunately, one is ultimately forced to consider nonlocal *and* nonlinear terms based on similar grounds, and such terms are indeed relevant below four dimensions. In some cases, local or global arclength conservation may be an important consideration in writing down a dynamics for the system. However, a local description is likely to be more correct in a more complicated system with screening effects (motion in a gel that screens hydrodynamic interactions) where a first principles approach becomes even more intractable. Therefore, this model is an important starting point towards understanding the scaling behavior of polymers under a uniform drift, a problem with great technological importance.

4. Ordering Phenomena on Growing Films

In many growth processes particles are highly mobile in an active layer at the surface, but are relatively immobile once incorporated in the bulk. We study models in which atoms are allowed to interact, equilibrate, and order on the surface, but are frozen in the bulk. Order parameter correlations in the resulting bulk material are highly anisotropic, reflecting its growth history. In a flat (layer by layer) growth mode, correlations perpendicular to the growth direction are similar to a two-dimensional system in equilibrium, while parallel correlations reflect the dynamics of such a system. When the growing film is rough, various couplings between height and order parameter fluctuations are possible. Such couplings modify the dynamic scaling properties of surface roughness, and may also change the critical behavior of the order parameter. Even the deterministic growth of the surface profile can result in interesting textures for the order parameter.

4.1. Introduction

For many technological applications, high quality films are grown by the process of vapor deposition. The properties of such films can be quite different from the same material produced in bulk equilibrium,^{55,56} reflecting their