

Preface

In spite of the title, the main subject of these lectures is not algebra, even less history, as one could conclude from a glance over the table of contents, but methodology. Their aim is to convey to the audience, which originally consisted of undergraduate students in mathematics, an idea of how mathematics is made. For such an ambitious project, the individual experience of any but the greatest mathematicians seems of little value, so I thought it appropriate to rely instead on the collective experience of generations of mathematicians, on the premise that there is a close analogy between collective and individual experience: the problems over which past mathematicians have stumbled are most likely to cause confusion to modern learners, and the methods which have been tried in the past are those which should come to mind naturally to the (gifted) students of today. The way in which *mathematics is made* is best learned from the way *mathematics has been made*, and that premise accounts for the historical perspective on which this work is based.

The theme used as an illustration for general methodology is the theory of equations. The main stages of its evolution, from its origins in ancient times to its completion by Galois around 1830 will be reviewed and discussed. For the purpose of these lectures, the theory of equations seemed like an ideal topic in several respects: first, it is completely elementary, requiring virtually no mathematical background for the statement of its problems, and yet it leads to profound ideas and to fundamental concepts of modern algebra. Secondly, it underwent a very long and eventful evolution, and several gems lie along the road, like Lagrange's 1770 paper, which brought order and method to the theory in a masterly way, and Vandermonde's visionary glimpse of the solution of certain equations of high degree, which hardly unveiled the principles of Galois theory sixty years

before Galois' memoir. Also instructive from a methodological point of view is the relationship between the general theory, as developed by Cardano, Tschirnhaus, Lagrange and Abel, and the attempts by Viète, de Moivre, Vandermonde and Gauss at significant examples, namely the so-called cyclotomic equations, which arise from the division of the circle into equal parts. Works in these two directions are closely intertwined like themes in a counterpoint, until their resolution in Galois' memoir. Finally, the algebraic theory of equations is now a closed subject, which reached complete maturity a long time ago; it is therefore possible to give a fair assessment of its various aspects. This is of course not true of Galois theory, which still provides inspiration for original research in numerous directions, but these lectures are concerned with the theory of equations and not with Galois theory of fields. The evolution from Galois' theory to modern Galois theory falls beyond the scope of this work; it would certainly fill another book like this one.

As a consequence of emphasis on historical evolution, the exposition of mathematical facts in these lectures is genetic rather than systematic, which means that it aims to retrace the concatenation of ideas by following (roughly) their chronological order of occurrence. Therefore, results which are logically close to each other may be scattered in different chapters, and some topics are discussed several times, by little touches, instead of being given a unique definitive account. The expected reward for these circumlocutions is that the reader could hopefully gain a better insight into the inner workings of the theory, which prompted it to evolve the way it did.

Of course, in order to avoid discussions that are too circuitous, the works of mathematicians of the past—especially the distant past—have been somewhat modernized as regards notation and terminology. Although considering sets of numbers and properties of such sets was clearly alien to the patterns of thinking until the nineteenth century, it would be futile to ignore the fact that (naive) set theory has now pervaded all levels of mathematical education. Therefore, free use will be made of the definitions of some basic algebraic structures such as field and group, at the expense of lessening some of the most original discoveries of Gauss, Abel and Galois. Except for those definitions and some elementary facts of linear algebra which are needed to clarify some proofs, the exposition is completely self-contained, as can be expected from a genetic treatment of an elementary topic.

It is fortunate to those who want to study the theory of equations that its long evolution is well documented: original works by Cardano, Viète, Descartes, Newton, Lagrange, Waring, Gauss, Ruffini, Abel, Galois are readily available through modern publications, some even in English translations. Besides these original

works and those of Girard, Cotes, Tschirnhaus and Vandermonde, I relied on several sources, mainly on Bourbaki's Note historique [6] for the general outline, on Van der Waerden's "Science Awakening" [62] for the ancient times and on Edwards' "Galois theory" [20] for the proofs of some propositions in Galois' memoir. For systematic expositions of Galois theory, with applications to the solution of algebraic equations by radicals, the reader can be referred to any of the fine existing accounts, such as Artin's classical booklet [2], Kaplansky's monograph [35], the books by Morandi [44], Rotman [50] or Stewart [56], or the relevant chapters of algebra textbooks by Cohn [14], Jacobson [33], [34] or Van der Waerden [61], and presumably to many others I am not aware of. In the present lectures, however, the reader will find a thorough treatment of cyclotomic equations after Gauss, of Abel's theorem on the impossibility of solving the general equation of degree 5 by radicals, and of the conditions for solvability of algebraic equations after Galois, with complete proofs. The point of view differs from the one in the quoted references in that it is strictly utilitarian, focusing (albeit to a lesser extent than the original papers) on the concrete problem at hand, which is to solve equations. Incidentally, it is striking to observe, in comparison, what kind of acrobatic tricks are needed to apply modern Galois theory to the solution of algebraic equations.

The exercises at the end of some chapters point to some extensions of the theory and occasionally provide the proof of some technical fact which is alluded to in the text. They are never indispensable for a good understanding of the text. Solutions to selected exercises are given at the end of the book.

This monograph is based on a course taught at the Université catholique de Louvain from 1978 to 1989, and was first published by Longman Scientific & Technical in 1988. It is a much expanded and completely revised version of my "Leçons sur la théorie des équations" published in 1980 by the (now vanished) Cabay editions in Louvain-la-Neuve. The wording of the Longman edition has been recast in a few places, but no major alteration has been made to the text.

I am greatly indebted to Francis Borceux, who invited me to give my first lectures in 1978, to the many students who endured them over the years, and to the readers who shared with me their views on the 1988 edition. Their valuable criticism and encouraging comments were all-important in my decision to prepare this new edition for publication. Through the various versions of this text, I was privileged to receive help from quite a few friends, in particular from Pasquale Mammone and Nicole Vast, who read parts of the manuscript, and from Murray Schacher and David Saltman, for advice on (American-) English usage. Hearty thanks to all of them. I owe special thanks also to T.S. Blyth, who edited the

manuscript of the Longman edition, to the staffs of the Centre général de Documentation (Université catholique de Louvain) and of the Bibliothèque Royale Albert 1^{er} (Brussels) for their helpfulness and for allowing me to reproduce parts of their books, and to Nicolas Rouche, who gave me access to the riches of his private library.

On the T_EXnical side, I am grateful to Suzanne D'Addato (who also typed the 1988 edition) and to Béatrice Van den Haute, and also to Camille Debiève for his help in drawing the figures.

Finally, my warmest thanks to Céline, Paul, Ève and Jean for their infectious joy of living and to Astrid for her patience and constant encouragement. The preparation of the 1988 edition for publication spanned the whole life of our little Paul. I wish to dedicate this book to his memory.