define $z = \eta(y)$ and use the model

$$z = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p + \epsilon,$$

where the error term represents not only the errors due to measurement or ignored factors, but also the error arising from the the above process of linearization. However, sometimes it may not be possible to linearize the model (see Exercise 1.13). □

The linearized model of this example is not directly used for fitting (1.4.4). As in the case of nonlinear regression, fitting of the generalized linear model often requires iterative methods. The linearized model may be fitted in order to produce reasonable initial iterates for such a procedure. See McCullagh and Nelder (1989) or Dobson (2001) for a detailed discussion of the generalized linear model and inference related to it.

1.5 Related models

The term *linear model* is sometimes used in a more general sense than what we consider here. From this broader perspective, any model which connects variables or their transformed versions through a linear relationship is a linear model. This description fits the generalized linear model, mentioned in the previous section. Additional examples are given below.

Example 1.5.1 (Autoregressive model for time series) When observations are collected sequentially at regular time intervals, the pattern of temporal dependence can be represented by the model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \epsilon_t, \quad t \text{ is an integer}, \qquad (1.5.1)$$

where x_t is the observation at time t, $\phi_1, \phi_2, \ldots, \phi_p$ are unspecified parameters, and ϵ_t is the unobservable model error at time t, which is assumed to be uncorrelated with the errors at other times. The errors are assumed to have an unspecified but constant variance. The form of the model (1.5.1) is very similar to (1.1.1). In fact, (1.5.1) can be viewed as a regression model where the explanatory variables are a collection of past values of the 'response' itself. Hence, it is known as the autoregressive model of order p, or simply AR(p). This model serves as a vehicle for *linear prediction*, that is, prediction of the future values of a variable in terms of a linear combination of its past values. The model has a wide range of applications, in areas such as economics, geology, speech and signal processing. □

A more general linear model for time series data is the autoregressive moving average model of order (p, q), also known as ARMA(p, q) and given by

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \theta_0 \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}, \quad (1.5.2)$$

where t is an integer.

Example 1.5.2 (State-space model for time series) Another important model for time series data is the state-space model, described by the pair of equations

$$\begin{aligned} \boldsymbol{x}_t &= \boldsymbol{B}_t \boldsymbol{x}_{t-1} + \boldsymbol{u}_t, \\ \boldsymbol{z}_t &= \boldsymbol{H}_t \boldsymbol{x}_t + \boldsymbol{v}_t, \end{aligned} \qquad t \text{ is an integer.} \qquad (1.5.3)$$

Here, $\boldsymbol{z}_t$ is a vector observed at time t (the *measurement vector*), $\boldsymbol{x}_t$ is an unobservable *state vector* at time t, $\boldsymbol{B}_t$ and $\boldsymbol{H}_t$ are known matrices and $\boldsymbol{u}_t$ and $\boldsymbol{v}_t$ are vectors of model errors at time t. The first of these two equations is referred to as the *state update equation* and the second one, the *measurement equation*. This model has many important applications and will be discussed further in Section 9.1.6. □

Example 1.5.3 (Mixed effects linear model) Suppose that we expand the model of Example 1.1.2 by including the effect of the various hospitals where the patients stay. The expanded model may be of the form

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u_1 \gamma_1 + \cdots + u_q \gamma_q + \epsilon,$$

where q is the total number of hospitals, and for $j = 1, 2, \ldots, q$, u_j is a binary variable indicating whether the jth hospital is involved and γ_j is the incremental contribution of the jth hospital to a hospital bill.

Note that unlike β_0, β_1 and β_2, the parameters $\gamma_1, \ldots, \gamma_q$ are themselves random. Determination of the extent of hospital-to-hospital variation in the bill amount — rather than the specific values of $\gamma_1, \ldots, \gamma_q$ — may be the object of a study. Since a combination of fixed and random parameters are involved, a model of this kind is called a *mixed effects linear model.* A general form of this model is discussed in Section 8.3.□

Example 1.5.4 (Simultaneous equations model) A model for the demand for food (d) is

$$d = \beta_0 + \beta_1 r + \beta_2 r_o + \beta_3 s + \epsilon, \tag{1.5.4}$$

where r is the price of food, r_o is the price of other commodities, s is the income and ϵ is an error term. On the other hand, the price r can be modelled as

$$r = \alpha_0 + \alpha_1 d + \alpha_2 w + \delta, \tag{1.5.5}$$

where w is an indicator of weather condition and δ is another error term. Note that both the equations are important for the study of demand and price of food, and the roles of response and explanatory variables are interchanged in the two equations. Models described by simultaneous linear equations of this kind are very important in econometrics. We shall not deal with such models in this book. The interested reader may refer to Fomby et al. (1984, Chapters 19–24). □

Inference related to the models described in this section has a close connection with that for the model (1.1.1). Several other issues such as model building, diagnostics, and validation are also common to all these 'linear models'. The model (1.1.1) — by virtue of being the simplest — serves as a common reference or benchmark for the other models.

1.6 Uses of the linear model

An important application of the linear model is in regression analysis where the average response is explained via other observed variables (called *regressors* in this context). Building a working model to describe the relationship among the variables is sometimes an end in itself. On