

Contents

Preface	xiii
I Introduction	1
1 Foreign exchange markets	3
1.1 Introduction	3
1.2 Historical background	4
1.3 Forex as an asset class	7
1.4 Spot forex	8
1.5 Derivatives: forwards, futures, calls, puts, and all that	9
1.6 References and further reading	17
II Mathematical preliminaries	19
2 Elements of probability theory	21
2.1 Introduction	21
2.2 Probability spaces	22
2.3 Random variables	26
2.4 Convergence of random variables and limit theorems	38
2.5 References and further reading	42
3 Discrete-time stochastic engines	43
3.1 Introduction	43
3.2 Time series	44
3.3 Binomial stochastic engines for single- and multi-period markets	46
3.4 Multinomial stochastic engines	55

3.5	References and further reading	57
4	Continuous-time stochastic engines	59
4.1	Introduction	59
4.2	Stochastic processes	61
4.3	Markov processes	63
4.4	Diffusions	65
4.5	Wiener processes	76
4.6	Poisson processes	81
4.7	SDE and Mappings	84
4.8	Linear SDEs	91
4.9	SDEs for jump-diffusions	96
4.10	Analytical solution of PDEs	98
4.10.1	Introduction	98
4.10.2	The reduction method	98
4.10.3	The Laplace transform method	103
4.10.4	The eigenfunction expansion method	104
4.11	Numerical solution of PDEs	106
4.11.1	Introduction	106
4.11.2	Explicit, implicit, and Crank-Nicolson schemes for solving one-dimensional problems	107
4.11.3	ADI scheme for solving two-dimensional problems	109
4.12	Numerical solution of SDEs	112
4.12.1	Introduction	112
4.12.2	Formulation of the problem	113
4.12.3	The Euler-Maruyama scheme	113
4.12.4	The Milstein scheme	114
4.13	References and further reading	116
III	Discrete-time models	119
5	Single-period markets	121
5.1	Introduction	121
5.2	Binomial markets with nonrisky investments	123
5.3	Binomial markets without nonrisky investments	140
5.4	General single-period markets	145
5.5	Economic constraints	147
5.6	Pricing of contingent claims	154
5.7	Elementary portfolio theory	162

5.8	The optimal investment problem	166
5.9	Elements of equilibrium theory	168
5.10	References and further reading	169
6	Multi-Period markets	171
6.1	Introduction	171
6.2	Stationary binomial markets	172
6.3	Non-stationary binomial markets	194
6.3.1	Introduction	194
6.3.2	The nonrecombining case	195
6.3.3	The recombining case	197
6.4	General multi-period markets	202
6.5	Contingent claims and their valuation and hedging	207
6.6	Portfolio theory	208
6.7	The optimal investment problem	210
6.8	References and further reading	212
IV	Continuous-time models	213
7	Stochastic dynamics of forex	215
7.1	Introduction	215
7.2	Two-country markets with deterministic investments	216
7.3	Two-country markets without deterministic investments	223
7.4	Multi-country markets	227
7.5	The nonlinear diffusion model	230
7.6	The jump diffusion model	232
7.7	The stochastic volatility model	233
7.8	The general forex evolution model	236
7.9	References and further reading	237
8	European options: the group-theoretical approach	239
8.1	Introduction	239
8.2	The two-country homogeneous problem, I	240
8.2.1	Formulation of the problem	240
8.2.2	Reductions of the pricing problem	245
8.2.3	Continuous hedging and the Greeks	248
8.3	Forwards, calls and puts	250
8.3.1	Definitions	250
8.3.2	Pricing via the Feynman-Kac formula	250

8.3.3	A naive pricing attempt	256
8.3.4	Pricing via the Fourier transform method	257
8.3.5	Pricing via the Laplace transform method	259
8.3.6	The limiting behavior of calls and puts	261
8.4	Contingent claims with arbitrary payoffs	263
8.4.1	Introduction	263
8.4.2	The decomposition formula	263
8.4.3	Call and put bets	264
8.4.4	Log contracts and modified log contracts	265
8.5	Dynamic asset allocation	266
8.6	The two-country homogeneous problem, II	275
8.7	The multi-country homogeneous problem	277
8.7.1	Introduction	277
8.7.2	The homogeneous pricing problem	278
8.7.3	Reductions	279
8.7.4	Probabilistic pricing and hedging	280
8.8	Some representative multi-factor options	281
8.8.1	Introduction	281
8.8.2	Outperformance options	282
8.8.3	Options on the maximum or minimum of several FXRs	284
8.8.4	Basket options	286
8.8.5	Index options	290
8.8.6	The multi-factor decomposition formula	291
8.9	References and further reading	292
9	European options, the classical approach	293
9.1	Introduction	293
9.2	The classical two-country pricing problem, I	294
9.2.1	The projection method	294
9.2.2	The classical method	296
9.2.3	The impact of the actual drift	297
9.3	Solution of the classical pricing problem	298
9.3.1	Nondimensionalization	298
9.3.2	Reductions	298
9.3.3	The pricing and hedging formulas for forwards, calls and puts	299
9.3.4	European options with exotic payoffs	306
9.4	The classical two-country pricing problem, II	310
9.5	The multi-country classical pricing problem	315
9.5.1	Introduction	315

9.5.2	Derivation	315
9.5.3	Reductions	315
9.5.4	Pricing and hedging of multi-factor options	317
9.6	References and further reading	317

10 Deviations from the Black-Scholes paradigm I: nonconstant volatility 319

10.1	Introduction	319
10.2	Volatility term structures and smiles	321
10.2.1	Introduction	321
10.2.2	The implied volatility	321
10.2.3	The local volatility	323
10.2.4	The inverse problem	325
10.2.5	How to deal with the smile	329
10.3	Pricing via implied t.p.d.f.'s	329
10.3.1	Implied t.p.d.f.'s and entropy maximization	329
10.3.2	Possible functional forms of t.p.d.f.'s	332
10.3.3	The chi-square pricing formula, I	335
10.3.4	The Edgeworth-type pricing formulas	338
10.4	The sticky-strike and the sticky-delta models	341
10.5	The general local volatility model	344
10.5.1	Introduction	344
10.5.2	Possible functional forms of local volatility	345
10.5.3	The hyperbolic volatility model	348
10.5.4	The displaced diffusion model	350
10.6	Asymptotic treatment of the local volatility model	353
10.7	The CEV model	359
10.7.1	Introduction	359
10.7.2	Reductions of the pricing problem	360
10.7.3	Evaluation of the t.p.d.f.	362
10.7.4	Derivative pricing	364
10.7.5	ATMF approximation	368
10.8	The jump diffusion model	371
10.8.1	Introduction	371
10.8.2	The pricing problem	371
10.8.3	Evaluation of the t.p.d.f.	372
10.8.4	Risk-neutral pricing	373
10.9	The stochastic volatility model	375
10.9.1	Introduction	375
10.9.2	Basic equations	376

10.9.3	Evaluation of the t.p.d.f.	379
10.9.4	The pricing formula	384
10.9.5	The case of zero correlation	386
10.10	Small volatility of volatility	387
10.10.1	Introduction	387
10.10.2	Basic equations	388
10.10.3	The martingale formulation	388
10.10.4	Perturbative expansion	389
10.10.5	Summary of ODEs	393
10.10.6	Solution of the leading order pricing problem	394
10.10.7	The square-root model	394
10.10.8	Computation of the implied volatility	397
10.11	Multi-factor problems	398
10.11.1	Introduction	398
10.11.2	The chi-square pricing formula, II	399
10.12	References and further reading	404
11	American Options	405
11.1	Introduction	405
11.2	General considerations	407
11.2.1	The early exercise constraint	407
11.2.2	The early exercise premium	408
11.2.3	Some representative examples	410
11.2.4	Rational bounds	411
11.2.5	Parity and symmetry	414
11.3	The risk-neutral valuation	415
11.4	Alternative formulations of the valuation problem	416
11.4.1	Introduction	416
11.4.2	The inhomogeneous Black-Scholes problem formulation	416
11.4.3	The linear complementarity formulation	418
11.4.4	The linear program formulation	420
11.5	Duality between puts and calls	421
11.6	Application of Duhamel's principle	422
11.6.1	The value of the early exercise premium	422
11.6.2	The location of the early exercise boundary	423
11.7	Asymptotic analysis of the pricing problem	425
11.7.1	Short-dated options	425
11.7.2	Long-dated and perpetual options	430
11.8	Approximate solution of the valuation problem	434
11.8.1	Introduction	434

11.8.2	Bermudan approximation and extrapolation to the limit	434
11.8.3	Quadratic approximation	440
11.9	Numerical solution of the pricing problem	442
11.9.1	Bermudan approximation	442
11.9.2	Linear complementarity	443
11.9.3	Integral equation	444
11.9.4	Monte-Carlo valuation	444
11.10	American options in a non-Black-Scholes framework	445
11.11	Multi-factor American options	445
11.11.1	Formulation	445
11.11.2	Two representative examples	446
11.12	References and further reading	449
12	Path-dependent options I: barrier options	451
12.1	Introduction	451
12.2	Single-factor, single-barrier options	452
12.2.1	Introduction	452
12.2.2	Pricing of single-barrier options via the method of images	453
12.2.3	Pricing of single-barrier options via the method of heat potentials	462
12.3	Static hedging	469
12.4	Single-factor, double-barrier options	472
12.4.1	Introduction	472
12.4.2	Formulation	473
12.4.3	The pricing problem without rebates	474
12.4.4	Pricing of no-rebate calls and puts and double-no-touch options	477
12.4.5	Pricing of calls and puts with rebate	482
12.5	Deviations from the Black-Scholes paradigm	484
12.5.1	Introduction	484
12.5.2	Barrier options in the presence of the term structure of volatility	484
12.5.3	Barrier options in the presence of constant elasticity of variance	486
12.5.4	Barrier options in the presence of stochastic volatility	492
12.6	Multi-factor barrier options	498
12.7	Options on one currency with barriers on the other currency	499
12.7.1	Introduction	499
12.7.2	Formulation	499
12.7.3	Solution via the Fourier method	501

12.7.4	Solution via the method of images	509
12.7.5	An alternative approach	513
12.8	Options with one barrier for each currency	514
12.8.1	General considerations	514
12.8.2	The Green's function	516
12.8.3	Two-factor, double-no-touch option	520
12.9	Four-barrier options	520
12.10	References and further reading	526
13	Path-dependent options II: lookback, Asian and other options	527
13.1	Introduction	527
13.2	Path-dependent options and augmented SDEs	528
13.2.1	Description of path dependent options	528
13.2.2	The augmentation procedure	534
13.2.3	The pricing problem for augmented SDEs	537
13.3	Risk-neutral valuation of path-dependent options	538
13.4	Probabilistic pricing	539
13.5	Lookback calls and puts	542
13.5.1	Description	542
13.5.2	Pricing via the method of images	543
13.5.3	Similarity reductions	547
13.5.4	Pricing via the Laplace transform	548
13.5.5	Probabilistic pricing	550
13.5.6	Barriers	552
13.6	Asian options	553
13.6.1	Description	553
13.6.2	Geometric averaging	553
13.6.3	Arithmetic averaging	556
13.6.4	Exact solution via similarity reductions	558
13.6.5	Pricing via the Laplace transform	561
13.6.6	Approximate pricing of Asian calls revisited	562
13.6.7	Discretely sampled Asian options	565
13.7	Timer, fader and Parisian options	566
13.7.1	Introduction	566
13.7.2	Timer options	566
13.7.3	Fader options	573
13.7.4	Parisian options	573
13.8	Standard passport options	578
13.8.1	Description	578
13.8.2	Similarity reductions and splitting	579

13.8.3	Pricing via the Laplace transform	581
13.8.4	Explicit solution for zero foreign interest rate	582
13.9	More general passport options	586
13.9.1	General considerations	586
13.9.2	Explicit solution for zero foreign interest rate	587
13.10	Variance and volatility swaps	591
13.10.1	Introduction	591
13.10.2	Description of swaps	591
13.10.3	Pricing and hedging of swaps via convexity adjustments	593
13.10.4	Log contracts and robust pricing and hedging of variance swaps	599
13.11	The impact of stochastic volatility on path-dependent options	602
13.11.1	The general valuation formula	602
13.11.2	Evaluation of the t.p.d.f.	603
13.11.3	A transformed valuation formula	608
13.12	Forward starting options (cliquets)	608
13.13	Options on volatility	611
13.13.1	The pricing problem	611
13.13.2	Pricing of variance swaps	613
13.13.3	Pricing of general swaps and swaptions	614
13.14	References and further reading	616
14	Deviations from the Black-Scholes paradigm II: market frictions	617
14.1	Introduction	617
14.2	Imperfect hedging	618
14.2.1	P&L distributions	618
14.2.2	Stop-loss start-gain hedging and local times	621
14.2.3	Parameter misspecification	622
14.3	The uncertain volatility model	627
14.4	Transaction costs	630
14.5	Liquidity risk	633
14.6	Default risk	635
14.6.1	Introduction	635
14.6.2	The pricing model	635
14.6.3	Pricing of defaultable European calls	637
14.6.4	Pricing of defaultable forward contracts	640
14.7	References and further reading	643

15 Future directions of research and conclusions	645
15.1 Introduction	645
15.2 Future directions	645
15.3 Conclusions	646
15.4 References and further reading	646