1. Introduction

Customarily, in water-wave propagation problems, distinction is made between regular and random waves. Regular waves typically consist of a few components (not necessarily harmonics of each other) while random waves consist of many components in which the phases are distributed randomly, usually uniformly. Transformation of waves take place as they propagate due to interactions between components, variation of the bottom and current and forcing conditions such as wind.

In wave motion, effect of shear is usually confined to a thin boundary layer. Experiences show that the important features of nonlinear wave interactions, refraction and shoaling due to bottom and current variation may be represented satisfactorily using a simpler mathematical description under the assumption of the fluid being inviscid and irrotational. This allows introduction of a wave potential $\Phi(\boldsymbol{x}, z, t)$ such that the velocity field $(\boldsymbol{u}, w)^T$ is given by $(\nabla\Phi, \partial\Phi/\partial z)^T$ where $\boldsymbol{u} = (u, v)^T = (u_1, u_2)^T$ is the horizontal velocity vector and w is the vertical component. We shall use in this text a coordinate system such that $\boldsymbol{x} = (x_1, x_2)^T = (x, y)^T$ is directed horizontally while z is directed upwards (opposite to the gravity acceleration vector). The still-water level and the sea bed are defined respectively by $z = 0$ and $z = -h(\boldsymbol{x})$. The governing equations for wave motion are then given by a field equation, the Laplace equation for Φ, a kinematic and a dynamic condition at the free surface $z = \zeta(\boldsymbol{x}, t)$ and a kinematic condition at the bottom,

$$\nabla^2\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0\,; \quad -h(\boldsymbol{x}, t) \le z \le \zeta(\boldsymbol{x}, t)\,, \tag{1a}$$

and the three boundary conditions,

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left\{(\nabla\Phi)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right\} + \frac{p}{\rho} + g\zeta = 0 \quad \text{at} \quad z = \zeta(\boldsymbol{x}, t)\,, \tag{1b}$$

$$\frac{\partial \zeta}{\partial t} + \nabla\Phi \cdot \nabla\zeta = \frac{\partial \Phi}{\partial z} \quad \text{at} \quad z = \zeta(\boldsymbol{x}, t)\,, \tag{1c}$$

and

$$\frac{\partial \Phi}{\partial t} + \nabla\Phi \cdot \nabla h = 0\,; \quad z = -h(\boldsymbol{x})\,, \tag{1d}$$

where

$$p = p_a - \mathcal{T}[\zeta] \quad \text{with} \quad \mathcal{T}[\zeta] = \gamma\nabla \cdot \left[\frac{\nabla\zeta}{(1 + |\nabla\zeta|^2)^{1/2}}\right]\,, \tag{1e}$$

γ being the surface tension. Both γ and the atmospheric pressure p_a are usually taken to be constant. Because the pressure in the water is reckoned with respect to the atmospheric pressure, p_a is taken to be zero for simplicity.

Taking $p_a = 0$ and $\gamma = 0$, we can eliminate ζ from the two free-surface conditions to get,

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} + \left[\frac{\partial}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla + \frac{1}{2}\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial z}\right]\left[\left(\frac{\partial \Phi}{\partial z}\right)^2 + |\nabla\Phi|^2\right] = 0 \quad \text{at} \quad z = \zeta\,. \tag{2}$$

For later use, we write the free-surface conditions of Eqs. (1c) and (1d) in terms of the free-surface wave potential $\varphi(\boldsymbol{x}, t) = \Phi\{\boldsymbol{x}, \zeta(\boldsymbol{x}, t), t\}$ and the free-surface vertical velocity $w^s = (\partial\Phi/\partial_z)|_{z=\zeta}$, e.g., see Dingemans (1997, Eqs. (64)),

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}(w^s)^2[1 + (\nabla\zeta)^2] + \frac{p}{\rho} + g\zeta = 0\,, \tag{3a}$$

$$\frac{\partial \zeta}{\partial t} + \nabla\varphi \cdot \nabla\zeta - w^s[1 + (\nabla\zeta)^2] = 0\,. \tag{3b}$$

While the framework described above is complete within the assumption of potential theory, tractable solutions of the exactly nonlinear equations are at the least computationally too heavy to permit computations over large area and gain insight into the physical processes. It is therefore essential to simplify the analytical setups with which the “relevant” physical processes may be quantified. One of the fundamental processes in wave propagation is "resonant" or "near-resonant" interaction between wave components. While for capillary waves (wave length λ less than 2 cm) on one end of the wave spectrum and shallow water waves ($\lambda > 7h$, h being the water depth) on the other, the significant interaction takes place between three waves (quadratic), the same occurs between four waves (cubic) over deep and intermediate water for gravity waves. A key assumption used in the simplification of this process is the narrow-bandedness of the spectrum. This allows one to represent the entire spectrum or components within a narrow-band by a single component with a fast-varying phase function (varying at a basic central frequency ω_0 and a corresponding wave number k_0) and a slow-modulation of the amplitude. Nonlinear transformations taking effect over a length scale much larger compared to $2\pi/k_0$ are accounted for in the slow-modulation of the amplitude function. Mathematical definitions and elaboration of the narrow-bandedness and the slow-modulation will be presented in the following section. We will first provide an insight to the nonlinear modulation followed by a derivation of the nonlinear (cubic) modulation equation, well-known as the nonlinear Schrödinger equation (NLS). This equation is the simplest form to study nonlinear modulation over deep and intermediate water. We will then discuss how the assumption of narrowness may be relaxed and higher-order description may be used. A summary of the experimental observations is compiled to discuss the "relevant" features during interactions and the need to go beyond NLS, the simplest form of the modulation equation. Of importance to coastal engineers over intermediate depth, we have paid attention to propagation over a varying bottom and in the presence of an ambient current. An example of the application of the nonlinear Schrödinger equation to the computation of the velocity field under waves has been given by Trulsen *et al.* (2001).

This article is by no means an exhaustive review of modulation phenomena. Capillary waves, except for a short discussion on their effects, are beyond the purview of this review. Similarly, shallow water waves are kept outside the present scope. For interested readers, additional references of previous reviews of modulation of water waves are by Yuen and Lake (1980, 1982), Hammack and Henderson (1993), and Dias and Kharif (1999).

2. Basic Insight into Modulational Processes

One of the earliest motivations for study of nonlinear modulation came from the observations of instabilities of water waves in wave flumes (Benjamin and