

may be quantified. One of the fundamental processes in wave propagation is “resonant” or “near-resonant” interaction between wave components. While for capillary waves (wave length λ less than 2 cm) on one end of the wave spectrum and shallow water waves ($\lambda > 7h$, h being the water depth) on the other, the significant interaction takes place between three waves (quadratic), the same occurs between four waves (cubic) over deep and intermediate water for gravity waves. A key assumption used in the simplification of this process is the narrow-bandedness of the spectrum. This allows one to represent the entire spectrum or components within a narrow-band by a single component with a fast-varying phase function (varying at a basic central frequency ω_0 and a corresponding wave number k_0) and a slow-modulation of the amplitude. Nonlinear transformations taking effect over a length scale much larger compared to $2\pi/k_0$ are accounted for in the slow-modulation of the amplitude function. Mathematical definitions and elaboration of the narrow-bandedness and the slow-modulation will be presented in the following section. We will first provide an insight to the nonlinear modulation followed by a derivation of the nonlinear (cubic) modulation equation, well-known as the nonlinear Schrödinger equation (NLS). This equation is the simplest form to study nonlinear modulation over deep and intermediate water. We will then discuss how the assumption of narrowness may be relaxed and higher-order description may be used. A summary of the experimental observations is compiled to discuss the “relevant” features during interactions and the need to go beyond NLS, the simplest form of the modulation equation. Of importance to coastal engineers over intermediate depth, we have paid attention to propagation over a varying bottom and in the presence of an ambient current. An example of the application of the nonlinear Schrödinger equation to the computation of the velocity field under waves has been given by Trulsen *et al.* (2001).

This article is by no means an exhaustive review of modulation phenomena. Capillary waves, except for a short discussion on their effects, are beyond the purview of this review. Similarly, shallow water waves are kept outside the present scope. For interested readers, additional references of previous reviews of modulation of water waves are by Yuen and Lake (1980, 1982), Hammack and Henderson (1993), and Dias and Kharif (1999).

2. Basic Insight into Modulational Processes

One of the earliest motivations for study of nonlinear modulation came from the observations of instabilities of water waves in wave flumes (Benjamin and

Feir, 1967). Although instability is a special feature of nonlinear modulation, we shall start with instability as an introduction. The essence of modulational instability can be treated in a number of factually equivalent ways. First, we consider an example based on the action equation and the equation of conservation of wave crests. Next, we consider the principal features of the Benjamin and Feir instability. Subsequently, we consider the nonlinear Schrödinger (NLS) equation of which we give an heuristic derivation. The Benjamin and Feir instability also results from a perturbation of the nonlinear Schrödinger equation.

A succinct physical explanation for the onset of instability of weakly nonlinear waves has been given by Lighthill (1965, 1967). We here follow his analysis,

- (1) Consider a pulse of weakly nonlinear waves in deep water which initially contains waves of uniform lengths.
- (2) Since the nonlinearity causes the crests of the waves with the larger amplitudes to travel more quickly ($\omega^2 = gk(1 + k^2 a^2)$), wave numbers tend to increase in the front of the pulse and decrease at the end of the pulse.
- (3) Because $dc_g/dk < 0$, the shorter waves in front of the pulse and the longer waves behind the pulse cause energy to approach the centre of the pulse, resulting in an increase of the amplitude in the centre of the pulse. This accelerates the instability.

For the case of finite-amplitude waves in water of finite depth, the spatial variation of the waves induces a mean flow and a change in mean water level. These mean flow and mean water level variations have a stabilizing effect because they cause wave numbers to increase behind the pulse and decrease in front of the pulse.

2.1. A simple example of instability

We first notice that, in absence of ambient currents, the modulation of a wave train can be described by the equation for wave action conservation and the equation for conservation of wave crests,

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{\partial \omega}{\partial k_j} a^2 \right) = 0, \tag{4a}$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0 \quad \text{with } i, j = 1, 2. \tag{4b}$$

An essential property of nonlinear behaviour is the dependence of the frequency ω on the amplitude a of the wave. For near-linear waves, ω may be written as:

$$\omega = \omega_0(\mathbf{k}) + \omega_2(\mathbf{k})a^2 + \dots \tag{5}$$

Simply using this expression for ω in Eq. (4b) results in:

$$\frac{\partial k_i}{\partial t} + \left(\frac{\partial \omega_0}{\partial k_j} + \frac{\partial \omega_2}{\partial k_j} a^2 \right) \frac{\partial k_j}{\partial x_i} + \omega_2(\mathbf{k}) \frac{\partial a^2}{\partial x_i} = 0. \tag{6}$$

The important term in Eq. (6) is $\omega_2 \partial a^2 / \partial x_i$. This term leads to a correction of $\mathcal{O}(a)$ in the characteristic velocities. The other extra term merely gives an $\mathcal{O}(a^2)$ correction of an existing term in $\partial k_j / \partial x_i$. Substitution of Eq. (5) into the wave action equation (4a) only results in extra terms of $\mathcal{O}(a^2)$. Neglecting this term and considering wave propagation in one spatial direction only, results into:

$$\frac{\partial a^2}{\partial t} + \omega'_0(k) \frac{\partial a^2}{\partial x} + \omega''_0(k) a^2 \frac{\partial k}{\partial x} = 0, \tag{7a}$$

$$\frac{\partial k}{\partial t} + \omega'_0(k) \frac{\partial k}{\partial x} + \omega_2(k) \frac{\partial a^2}{\partial x} = 0, \tag{7b}$$

where a prime denotes differentiation to k . The characteristics (or wave rays) of this set are given by:

$$\frac{dx}{dt} = \omega'_0(k) \mp \sqrt{\omega_2(k) \omega''_0(k)} a. \tag{8}$$

If $\omega_2(k) \omega''_0(k) > 0$, we have two different characteristic velocities. This is clearly a nonlinear effect because for vanishing a (the linear approximation), we have $dx/dt = \omega'_0(k)$, as follows directly from Eq. (4b) when $\omega = \omega_0$ is used.

If $\omega_2(k) \omega''_0(k) < 0$, the characteristic velocities of Eq. (8) are complex and the system in Eqs. (7a) and (7b) is elliptic. That means small sub-harmonic modulations will grow with time and that the wave system is unstable in this sense. For waves on deep water, the dispersion relation reads,

$$\omega^2 = gk(1 + (ak)^2 + \dots), \tag{9a}$$

or

$$\omega = \sqrt{gk} \left(1 + \frac{1}{2}(ak)^2 + \dots \right). \tag{9b}$$

We thus have $\omega_0 = \sqrt{gk}$ and $\omega_2 = g^{1/2}k^{5/2}/2$ and we have $\omega_2\omega_0'' = -gk/8 < 0$ for all k for waves on deep water.

2.2. Basic ideas of the Benjamin–Feir instability mechanism

A Stokes wave can be unstable due to side-band perturbations as shown by Benjamin and Feir (1967). The essential steps are as follows (see also Stuart and DiPrima, 1978),

- (1) Consider the small-amplitude wave,

$$a \exp[i(kx - \omega t)]. \tag{10}$$

- (2) Nonlinear interactions force the generation of harmonics of this mode in particular the second harmonic which is proportional to:

$$a^2 \exp[2i(kx - \omega t)]. \tag{11}$$

- (3) Suppose that two modal perturbations arise,

$$\text{An upper side band } a_1 \exp[i(k_1x - \omega_1t)], \tag{12a}$$

and

$$\text{a lower side band } a_2 \exp[i(k_2x - \omega_2t)], \tag{12b}$$

with $a_1, a_2 \ll a$.

- (4) Nonlinear interaction between the second harmonic and these side-band perturbations produces,

$$a^2 a_1 \exp[i(2k - k_1)x - i(2\omega - \omega_1)t], \tag{13a}$$

and

$$a^2 a_2 \exp[i(2k - k_2)x - i(2\omega - \omega_2)t], \tag{13b}$$

and also the sum-interaction terms.

- (5) Suppose now that:

$$k_1 + k_2 = 2k, \quad \omega_1 + \omega_2 = 2\omega. \tag{14}$$

Then

$$a^2 a_2 \exp[i(k_1x - \omega_1t)] \sim \text{upper side band}, \tag{15a}$$

and

$$a^2 a_1 \exp[i(k_2x - \omega_2t)] \sim \text{lower side band}. \tag{15b}$$

We see that the *simultaneous* presence of the upper and lower side bands in association with the second harmonic results in a mutual reinforcement of resonance. The instability mechanism for Stokes waves is essentially the exponential growth in time resulting from this synchronous resonance.

Mathematical features of modulational instability will be discussed later in section 8.1.

3. Nonlinear Schrödinger-type Equations: Horizontal Bottom

The nonlinear Schrödinger (NLS) equation is the simplest example of an evolution equation for weakly nonlinear waves with strong frequency dispersion. The NLS equation describes the nonlinear evolution of a wave group with carrier wave number k and frequency ω . We first consider the properties of a wave group (see also Chu and Mei, 1971). Let the group consist of a superposition of two sinusoidal waves with amplitude a_0 and different (ω_1, k_1) and (ω_2, k_2) . The resulting wave is written as $a \cos \chi$ with:

$$a(x, t) = 2a_0 \cos \left(\frac{\delta k}{2}x - \frac{\delta \omega}{2}t \right), \quad (16)$$

$$\chi(x, t) = \frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t, \quad (17)$$

with $\delta k = k_1 - k_2$ and $\delta \omega = \omega_1 - \omega_2$. We now introduce the carrier wave number and frequency by $k = \partial \chi / \partial x = (k_1 + k_2)/2$ and $\omega = -\partial \chi / \partial t = (\omega_1 + \omega_2)/2$. It is seen now that when the individual waves satisfy the dispersion relation $\omega_j = \Omega(k_j)$, it is not true for $\omega = \Omega(k)$ except for nondispersive waves. Indeed, one has $\omega = (\Omega(k_1) + \Omega(k_2))/2$ which becomes Taylor expansion for small $\delta k/k$,

$$\omega = \Omega(k) + \frac{(\delta k)^2}{8} \frac{\partial^2 \Omega(k)}{\partial k^2} + \frac{(\delta k)^4}{384} \frac{\partial^4 \Omega(k)}{\partial k^4} + \dots \quad (18)$$

3.1. A heuristic derivation of the NLS equation

A heuristic derivation of the NLS equation has been given by a number of authors, amongst which are Karpman and Krushkal' (1969), Kadomtsev and Karpman (1971), Karpman (1975, section 27), Jeffrey and Kawahara (1982, p. 59), Yuen and Lake (1982, p. 75), and Dingemans (1997, section 8.3.2). The derivation starts with a harmonic wave with basic frequency and wave number