We see that the *simultaneous* presence of the upper and lower side bands in association with the second harmonic results in a mutual reinforcement of resonance. The instability mechanism for Stokes waves is essentially the exponential growth in time resulting from this synchronous resonance.

Mathematical features of modulational instability will be discussed later in section 8.1.

3. Nonlinear Schrödinger-type Equations: Horizontal Bottom

The nonlinear Schrödinger (NLS) equation is the simplest example of an evolution equation for weakly nonlinear waves with strong frequency dispersion. The NLS equation describes the nonlinear evolution of a wave group with carrier wave number k and frequency ω. We first consider the properties of a wave group (see also Chu and Mei, 1971). Let the group consist of a superposition of two sinusoidal waves with amplitude a_0 and different (ω_1, k_1) and (ω_2, k_2). The resulting wave is written as $a \cos \chi$ with:

$$a(x,t) = 2a_0 \cos\left(\frac{\delta k}{2}x - \frac{\delta\omega}{2}t\right), \tag{16}$$

$$\chi(x,t) = \frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\,, \tag{17}$$

with $\delta k = k_1 - k_2$ and $\delta\omega = \omega_1 - \omega_2$. We now introduce the carrier wave number and frequency by $k = \partial\chi/\partial x = (k_1 + k_2)/2$ and $\omega = -\partial\chi/\partial t = (\omega_1 + \omega_2)/2$. It is seen now that when the individual waves satisfy the dispersion relation $\omega_j = \Omega(k_j)$, it is not true for $\omega = \Omega(k)$ except for nondispersive waves. Indeed, one has $\omega = (\Omega(k_1) + \Omega(k_2))/2$ which becomes Taylor expansion for small $\delta k/k$,

$$\omega = \Omega(k) + \frac{(\delta k)^2}{8}\frac{\partial^2\Omega(k)}{\partial k^2} + \frac{(\delta k)^4}{384}\frac{\partial^4\Omega(k)}{\partial k^4} + \cdots\,. \tag{18}$$

3.1. *A heuristic derivation of the NLS equation*

A heuristic derivation of the NLS equation has been given by a number of authors, amongst which are Karpman and Krushkal' (1969), Kadomtsev and Karpman (1971), Karpman (1975, section 27), Jeffrey and Kawahara (1982, p. 59), Yuen and Lake (1982, p. 75), and Dingemans (1997, section 8.3.2). The derivation starts with a harmonic wave with basic frequency and wave number given as (ω_0, k_0),

$$\zeta(x,t) = \mathrm{Re}\{a(x,t)\mathrm{e}^{i(k_0x-\omega_0t)+\gamma(x,t)}\} = \mathrm{Re}\{A^*(x,t)\mathrm{e}^{i\gamma(x,t)}\}$$
$$= \mathrm{Re}\{A(x,t)\mathrm{e}^{-i\gamma(x,t)}\}\,. \tag{19}$$

The dispersion relation is written in the general form,

$$\omega = \Omega(k, a^2)\,. \tag{20}$$

As we consider modulated waves, we now focus on small changes in the carrier wave frequency and wave number ω_0 and k_0: $\omega = \omega_0 + \delta\omega$ and $k = k_0 + \delta k$ with $\delta\omega/\omega_0 = \mathcal{O}(\beta) \ll 1$ and $\delta k/k_0 = \mathcal{O}(\beta)$. We now consider small changes of this basic state, i.e., we expand around the basic state consisting of (ω_0, k_0) and zero amplitude (thus around the linear approximation). As amplitude measure, we use the small parameter ε which stands for either ak or a/h whichever parameter is the most restrictive. Then, we have,

$$\omega = \Omega(k_0,0) + \left(\frac{\partial\Omega}{\partial k}\right)_0 \delta k + \frac{1}{2}\left(\frac{\partial^2\Omega}{\partial k^2}\right)_0 (\delta k)^2 + \left(\frac{\partial\Omega}{\partial a^2}\right)_0 a^2 + o(\beta^2, \varepsilon^2)\,, \tag{21}$$

with $()_0$ denoting that the quantity is evaluated for the basic state (k_0 and $a = 0$). We introduce the abbreviations,

$$\omega_0 = \Omega(k_0, 0)\,, \quad c_g = \left(\frac{\partial\Omega}{\partial k}\right)_0 \quad \text{and} \quad \omega_0'' = \left(\frac{\partial^2\Omega}{\partial k^2}\right)_0\,, \tag{22}$$

so that Eq. (21) may be written as:

$$-(\omega - \omega_0) + c_g\delta k + \frac{1}{2}\omega_0''(\delta k)^2 + \left(\frac{\partial\Omega}{\partial a^2}\right) a^2 + \cdots = 0\,. \tag{23}$$

We now introduce operator correspondence for which we have $\delta k \to i\partial_X$ and $\omega - \omega_0 = \delta\omega \to -i\partial_T$ with X is the slow spatial coordinate $X = \beta x$ and T the slow time $T = \beta t$. Operating on A^* results in:

$$i\left(\frac{\partial}{\partial T} + c_g\frac{\partial}{\partial X}\right)A^* - \frac{1}{2}\omega_0''\frac{\partial^2 A^*}{\partial X^2} + \left(\frac{\partial\Omega}{\partial|A|^2}\right)_0 |A|^2A^* = 0\,. \tag{24}$$

Introduction of the moving coordinate ξ and the time variable τ by (see also next subsection):

$$\xi = X - c_gT \quad \text{and} \quad \tau = \varepsilon T\,, \tag{25}$$

and changing to A leads to the NLS equation in its usual form,

$$i\frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} - \nu_1 |A|^2 A = 0\,, \tag{26}$$

where the abbreviations,

$$\lambda_1 = \frac{1}{2}\omega_0'' \quad \text{and} \quad \nu_1 = \left(\frac{\partial \Omega}{\partial |A|^2}\right)_0\,, \tag{27}$$

have been used and where $\varepsilon = \beta$ has been set. For deep water we have,

$$\omega = \left[1 + \frac{1}{2}k^2a^2\right]\sqrt{gk} = \left[1 + \frac{1}{2}k^2a^2\right]\omega_0\,, \tag{28}$$

and we thus obtain for deep water the NLS equation,

$$i\frac{\partial A}{\partial \tau} - \frac{1}{8}\frac{\omega_0}{k^2}\frac{\partial^2 A}{\partial \xi^2} - \frac{1}{2}\omega_0 k^2 |A|^2 A = 0\,. \tag{29}$$

3.2. *The scaling in the NLS equation*

The NLS equation reads,

$$i\frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} - \nu_1 |A|^2 A = 0\,, \tag{30a}$$

where

$$\xi = \beta(x - c_g t) \quad \text{and} \quad \tau = \varepsilon\beta t\,. \tag{30b}$$

In order to see why τ and ξ are scaled as Eq. (30b), we follow simple arguments of Asano (1974). Consider two waves with frequency and wave numbers (ω, k) and (ω', k'). If the differences $\Omega = \omega' - \omega$ and $K = k' - k$ are small, then, the resulting wave will have a long-wave envelope with wave number K and frequency Ω. The dispersion relation is:

$$\Omega = {c_g}^{(0)}K + \frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}K^2 + \frac{1}{6}\frac{\partial^3 \omega}{\partial k^3}K^3 + \cdots\,, \tag{31}$$

where ${c_g}^{(0)} = \partial\omega/\partial k$ is the group velocity of the carrier wave (ω, k). Then, the linear phase velocity of the wave envelope, $V = \omega/K$ is:

$$V = {c_g}^{(0)} + \frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}K + \frac{1}{6}\frac{\partial^3 \omega}{\partial k^3}K^2 + \cdots\,. \tag{32}$$

On the other hand, considering the long wave with (Ω, K) to be a nonlinear long wave (i.e., without frequency dispersion), its propagation velocity may be described by the characteristic velocity dx/dt which expanded in powers of ε, can be written as:

$$\frac{dx}{dt} = {c_g}^{(0)} + \varepsilon {c_g}^{(1)} + \varepsilon^2 {c_g}^{(2)} + \cdots . \tag{33}$$

The coupling between the modulation and the nonlinearity is strongest when both effects are of equal order of magnitude. Thus, $K = \mathcal{O}(\varepsilon)$. Because we already had $K = \mathcal{O}(\beta)$, this means that $\beta \sim \varepsilon$. Notice that this is similar to the case of stationary fairly long waves where the linear dispersive long wave velocity is $c \cong \sqrt{gh}\{1 - (kh)^2/3 + \cdots\}$ and the nonlinear nondispersive long wave velocity is $c \cong \sqrt{gh}\{1 + a/(2h) + \cdots\}$. For permanency, it was required that $(kh)^2 \sim a/h$.

Because $K = \mathcal{O}(\beta)$ is measured with respect to the carrier wave number k, the scale on which the wave group has to be considered is $\Lambda = \lambda/\beta$, and so $X = \beta x$ where x is scaled with $\lambda = 2\pi/k$. In order to remain near the centre of the wave group, a moving coordinate system should be applied. The coordinate along the characteristic curve is then $\xi = \beta(x - {c_g}^{(0)}t)$. Because one has $x = \beta^{-1}\xi + {c_g}^{(0)}t$, it follows that $dx/dt = \beta^{-1}d\xi/dt + {c_g}^{(0)}$. From Eq. (33), one obtains $dx/dt = \varepsilon {c_g}^{(1)} + {c_g}^{(0)}$. In order that these expressions are the same, it is necessary that:

$$\frac{1}{\beta\varepsilon}\frac{d\xi}{dt} = {c_g}^{(1)} .$$

Introducing the slow time scale $\tau = \varepsilon\beta t$, one obtains $d\xi/dt = {c_g}^{(1)}(\xi, \tau)$. This produces,

$$\xi = \beta(x - {c_g}^{(0)}t) , \quad \tau = \varepsilon\beta t . \tag{34}$$

3.3. *A sketch of the derivation in two horizontal dimensions*

We will give here a sketch of the derivation of the NLS-type of equations in two horizontal dimensions for the case of a horizontal bottom. This derivation follows Davey and Stewartson (1974).

At time $t = 0$, the free-surface elevation is given as:

$$\zeta(\boldsymbol{x}, 0) = i\varepsilon\frac{\omega}{g}a(\beta x_1, \beta x_2)\mathrm{e}^{ikx_1} + CC , \tag{35}$$

which thus represents a progressive wave in the x_1 direction with a slowly-varying amplitude. The amplitude a is measured here in $\mathrm{m}^2\mathrm{s}^{-1}$, i.e., a is a measure for the amplitude of the velocity potential $\Phi(\boldsymbol{x}, z, t)$. The governing equations are the Laplace equation for Φ in $-h < z < \zeta(\boldsymbol{x}, t)$, the kinematic conditions at $z = -h$ and $z = \zeta$, and the dynamic condition at $z = \zeta$. A solution of these equations is taken to be of the following form,

$$\zeta(\boldsymbol{x}, t) = \sum_{m=-\infty}^{\infty} \zeta_m E^m \,, \quad \Phi(\boldsymbol{x}, z, t) = \sum_{m=-\infty}^{\infty} \phi_m E^m \,, \tag{36a}$$

with

$$\zeta_{-m} = \zeta_m^* \,, \quad \phi_{-m}^* = \phi_m \,, \quad E = \exp[i(kx_1 - \omega t)] \,. \tag{36b}$$

Subsequently, the functions $\zeta_m(\boldsymbol{x}, t)$ and $\phi_m(\boldsymbol{x}, t)$ are expanded as:

$$\zeta_m = \sum_{n=m}^{\infty} \varepsilon^n \zeta^{(n,m)}(\xi, \eta, \tau) \,, \tag{37a}$$

$$\phi_m = \sum_{n=m}^{\infty} \varepsilon^n \phi^{(n,m)}(\xi, \eta, z, \tau) \,, \tag{37b}$$

$$\xi = \varepsilon(x_1 - c_g t) \,, \quad \eta = \varepsilon x_2 \,, \quad \tau = \varepsilon^2 t \,, \quad (\varepsilon = \beta) \,, \tag{37c}$$

where c_g is the linear group velocity and no distinction between the two scales ε and β is made anymore.

The zeroth-order, zeroth-harmonic terms $\zeta^{(0,0)}$ and $\phi^{(0,0)}$ are taken to be zero, $\zeta^{(0,0)} = \phi^{(0,0)} = 0$. Upon substitution of the representation for Φ in the Laplace equation, we get,

$$\sum_m \sum_n \varepsilon^n E^m \left[\frac{\partial^2 \phi^{(n,m)}}{\partial z^2} - k^2 m^2 \phi^{(n,m)} + 2i\varepsilon k m \frac{\partial \phi^{(n,m)}}{\partial \xi} \right.$$
$$\left. + \varepsilon^2 \frac{\partial^2 \phi^{(n,m)}}{\partial \xi^2} + \varepsilon^2 \frac{\partial^2 \phi^{(n,m)}}{\partial \eta^2} \right] = 0 \,. \tag{38}$$

Using the bottom condition, it can be seen that $\phi^{(1,0)}$ and $\phi^{(2,0)}$ are independent of z. $\phi^{(3,0)}$ is the first zeroth-harmonic term which depends on z,

$$\frac{\partial \phi^{(3,0)}}{\partial z} = -(z + h) \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \phi^{(1,0)} \,.$$

For $\phi^{(1,1)}$ and $\phi^{(2,2)}$, the following expressions are found,

$$\phi^{(1,1)} = B(\xi,\eta,\tau)\frac{\cosh k(z+h)}{\cosh kh}\,, \quad \phi^{(2,2)} = F(\xi,\eta,\tau)\frac{\cosh 2k(z+h)}{\cosh 2kh}\,, \tag{39a}$$

and for $\phi^{(2,1)}$ is found,

$$\begin{aligned}\phi^{(2,1)} &= D(\xi,\eta,\tau)\frac{\cosh k(z+h)}{\cosh kh} \\ &\quad - i\frac{\partial B}{\partial \xi}\frac{(z+h)\sinh k(z+h) - h\tanh kh\cosh k(z+h)}{\cosh kh}\,. \end{aligned} \tag{39b}$$

Subsequently, the expressions for ζ and Φ are substituted in the kinematic and dynamic free-surface conditions. The coefficients of $\varepsilon^n E^m$ are equated to zero for $n = 1, 2, 3$ and $m = 0, 1, 2$ and then it is possible to calculate the quantities $\zeta^{(n,m)}$ for $n = 1, 2$ and $m = 0, 1, 2$. For $n = 3$, the component $\partial\phi^{(3,0)}/\partial z$ of the vertical velocity gives a contribution in the kinematic free surface condition. Eliminating $\zeta^{(2,0)}$, a differential equation for $\phi^{(1,0)}$ is found,

$$(gh - {c_g}^2)\frac{\partial^2\phi^{(1,0)}}{\partial\xi^2} + gh\frac{\partial^2\phi^{(1,0)}}{\partial\eta^2} = -k^2[2c + c_g(1-\sigma^2)]\frac{\partial|B|^2}{\partial\xi}\,, \tag{40}$$

where c is the phase velocity, $c = \omega/k$ and $\sigma = \tanh kh$. Also, an equation for $\phi^{(3,0)}$ follows, but, that one is not needed here.

The coefficients of $\varepsilon^3 E^1$ in the kinematic and dynamic free-surface conditions yield two equations for $\phi^{(3,1)}$ and $\zeta^{(3,1)}$ at $z = 0$. Elimination of one of these from the two equations shows that the equations are only compatible if the following condition is fulfilled,

$$\begin{aligned} 2i\omega\frac{\partial B}{\partial\tau} + \omega\frac{d^2\omega}{dk^2}\frac{\partial^2 B}{\partial\xi^2} + cc_g\frac{\partial^2 B}{\partial\eta^2} &= \frac{1}{2}k^4(9\sigma^{-2} - 12 + 13\sigma^2 - 2\sigma^4)|B|^2 B \\ &\quad + k^2(2c + c_g(1-\sigma^2))B\frac{\partial\phi^{(1,0)}}{\partial\xi}\,. \end{aligned} \tag{41}$$

It is noted that Eqs. (40) and (41) describe the evolution of the progressive wave to first order in ε. $\phi^{(1,0)}$ is the first-order mean current and B is the amplitude of $\phi^{(1,1)}$ evaluated at $z = 0$. Equation (40) is a Poisson-type of equation for the mean flow with forcing and Eq. (41) is a NLS equation, coupled to the mean flow.

For later use, these equations are written in the following form,

$$i\frac{\partial B}{\partial \tau} + \lambda_1 \frac{\partial^2 B}{\partial \xi^2} + \mu_1 \frac{\partial^2 B}{\partial \eta^2} = \chi_1^\phi |B|^2 B + \chi_2 B \frac{\partial \phi^{(1,0)}}{\partial \xi}\,, \tag{42a}$$

$$\alpha \frac{\partial^2 \phi^{(1,0)}}{\partial \xi^2} + gh \frac{\partial^2 \phi^{(1,0)}}{\partial \eta^2} = -\kappa_2^\phi \frac{\partial}{\partial \xi}(|B|^2)\,, \tag{42b}$$

with the coefficients given by:

$$\chi_1^\phi = \frac{k^4}{4\omega}(9\sigma^{-2} - 12 + 13\sigma^2 - 2\sigma^4)\,, \tag{43a}$$

$$\chi_2 = \frac{k^2}{2\omega}[2c + c_g(1-\sigma^2)]\,, \tag{43b}$$

$$\alpha = gh - {c_g}^2\,, \tag{43c}$$

$$\kappa_2^\phi = k^2[2c + c_g(1-\sigma^2)]\,, \tag{43d}$$

$$\lambda_1 = \frac{1}{2\omega}[-{c_g}^2 + gh(1-\sigma^2)(1-kh\sigma)]\,, \tag{43e}$$

$$\mu_1 = \frac{\omega'}{2k} \equiv \frac{c_g}{2k} \geq 0\,. \tag{43f}$$

Because B is the envelope of the wave potential, we write χ_1^ϕ and κ_2^ϕ to remind one of the fact that these coefficients depend on the definition of B being the envelope of the wave potential or the free-surface elevation; in the latter case, we write χ_1^ζ and κ_2^ζ. We drop the superscript in those situations in which the value of the coefficients is not immediately needed. We have,

$$\chi_1^\phi = \frac{\omega^2}{g^2}\chi_1^\zeta \quad \text{and} \quad \kappa_2^\phi = \frac{\omega}{g}\kappa_2^\zeta\,. \tag{44}$$

The solutions for the $\zeta^{(n,m)}$ follow from the free-surface conditions for the various orders of approximation using the solutions of Eq. (39). We obtain (Davey and Stewartson, 1974),

$$g\zeta^{(1,1)} = i\omega B\,, \quad g\zeta^{(2,0)} = c_g \frac{\partial \phi^{(1,0)}}{\partial \xi} - k^2(1-\sigma^2)|B|^2\,, \tag{45a}$$

$$g\zeta^{(2,1)} = i\omega D + c_g \frac{\partial B}{\partial \xi}\,, \quad g\zeta^{(2,2)} = k^2 B^2 \left(\frac{\sigma^2 - 3}{2\sigma^2}\right)\,, \tag{45b}$$

$$\omega F = 3ik^2B^2\left(\frac{1-\sigma^4}{4\sigma^2}\right). \tag{45c}$$

3.3.1. *An alternative 2DH set of equations*

A quantity $Q(\xi, \eta, \tau)$ is introduced by:

$$Q(\xi,\eta,\tau) = \frac{c_g}{k^2}\frac{\partial\phi^{(1,0)}}{\partial\xi} + \frac{1}{gh-{c_g}^2}{c_g}^2(1-\sigma^2)|B|^2\,. \tag{46}$$

The mean surface elevation $\zeta^{(2,0)}$ then is given by:

$$\zeta^{(2,0)} = \frac{k^2}{g}Q - \frac{k[\sigma+2kh(1-\sigma^2)]}{gh-{c_g}^2}|B|^2\,. \tag{47}$$

Equations (41) and (40) can then be written in the form,

$$i\frac{\partial B}{\partial\tau} + \lambda_1\frac{\partial^2 B}{\partial\xi^2} + \mu_1^\phi\frac{\partial^2 B}{\partial\eta^2} = \nu_1^\phi|B|^2B + \nu_2 QB\,, \tag{48a}$$

$$(gh-{c_g}^2)\frac{\partial^2 Q}{\partial\xi^2} + gh\frac{\partial^2 Q}{\partial\eta^2} = \kappa_1^\phi\frac{\partial^2|B|^2}{\partial\eta^2}\,, \tag{48b}$$

with

$$\lambda_1 = \frac{1}{2}\omega''(k) \le 0\,,\quad \mu_1 = \frac{\omega'(k)}{2k} \equiv \frac{c_g}{2k} \ge 0\,, \tag{49a}$$

$$\begin{aligned}\nu_1^\phi = \frac{k^4}{4\omega\sigma^2}\Bigg[&9-10\sigma^2+9\sigma^4\\ &-\frac{2\sigma^2}{gh-{c_g}^2}\{4c^2+4cc_g(1-\sigma^2)+gh(1-\sigma^2)^2\}\Bigg]\,,\end{aligned} \tag{49b}$$

$$\nu_2 = \frac{k^4}{2\omega c_g}[2c+c_g(1-\sigma^2)] \ge 0\,, \tag{49c}$$

$$\kappa_1^\phi = ghc_g\frac{2c+c_g(1-\sigma^2)}{gh-{c_g}^2}\,. \tag{49d}$$

Equations (48) are sometimes referred to as the Davey and Stewartson equations (D&S Eqs.). In fact, it would be more appropriate to call these and similar equations the Benney and Newell equations because they were first derived for water waves in the paper of Benney and Newell (1967). It is noticed that, in absence of capillarity, one has $gh - {c_g}^2 > 0$ for waves of finite length and therefore, Q satisfies an equation of Poisson type (i.e., an elliptic equation). We also note that Eqs. (48) are derived under the conditions that $\varepsilon = ka$, $\beta = \lambda/\Lambda = \Delta k/k$ and $\Delta k_2/\Delta k_1$ are all small parameters of the same order where $k = (k_1^2 + k_2^2)^{1/2}$.

3.4. *Conservation laws*

It is well known that the NLS has a infinite set of conservation laws. Writing the NLS equation as:

$$i\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + \sigma_1 |q|^2 q = 0 \quad \text{in} \quad -\infty < x < \infty\,, \quad t \geq 0\,, \tag{50}$$

Zakharov and Shabat (1972) (see also Lamb, 1980, p. 111) gave the conserved quantities in the form,

$$(2i)^n C_n = \int_{-\infty}^{\infty} f_n(x)dx\,, \quad n \geq 1\,, \tag{51a}$$

with

$$f_{n+1} = q\frac{d}{dx}\left(\frac{f_n}{q}\right) + \sum_{j+k=n} f_j f_k\,, \quad f_1 = \frac{1}{2}\sigma_1 |q|^2\,, \tag{51b}$$

and $f_0 \equiv 0$. The first two conservation laws are readily obtained as follows (e.g., see Debnath, 1994, p. 349). The first conservation law is obtained by multiplying Eq. (50) by $\bar{q}_x$ and the complex conjugate equation by q_x and subtracting these equations. An integration to x then leads to $d/dt \int_{-\infty}^{\infty} |q|^2 dx = 0$.

The second conserved quantity f_2 is obtained as follows. First, we multiply Eq. (50) with $\bar{q}_x$ and its complex conjugate with q_x and add these equations to give,

$$i(q_t\bar{q}_x - \bar{q}_t q_x) + (q_{xx}\bar{q}_x + \bar{q}_{xx}q_x) + \sigma_1 |q|^2 (q\bar{q}_x + \bar{q}q_x) = 0\,. \tag{52}$$

Secondly, Eq. (50) and its complex conjugate are differentiated to x and we multiply the resulting equations with $\bar{q}$ and q respectively. Adding these equations leads to:

$$i(\bar{q}q_{xt} - q\bar{q}_{xt}) + (q_{xxx}\bar{q} + \bar{q}_{xxx}q) + \sigma_1[\bar{q}(|q|^2 q)_x + q(|q|^2\bar{q})_x] = 0\,. \tag{53}$$

Subtraction of Eq. (53) from Eq. (52) and integration to x then leads to:

$$\frac{d}{dt}\int_{-\infty}^{\infty} i(\bar{q}q_x - q\bar{q}_x)dx = 0\,. \tag{54}$$

Usually, real conserved quantities are used with different numerical factors, see, e.g., Watanabe *et al.* (1979), also given in Dingemans (1997, p. 928). The first two are:

$$I_1 = \int_{-\infty}^{\infty} dx|q(x,t)|^2 dx \quad \text{and} \quad I_2 = \frac{1}{2i}\int_{-\infty}^{\infty}(q^* q_x - q_x^* q)dx\,. \tag{55}$$

Watanabe *et al.* (1979) gave also a different conserved quantity I_0 as:

$$I_0 = \int_{-\infty}^{\infty}\left\{x|q|^2 - \frac{t}{2i}(q^* q_x - q q_x^*)\right\}dx\,, \tag{56}$$

which upon differentiation to t leads to the relation,

$$\frac{d}{dt}\int_{-\infty}^{\infty} x|q(x,t)|^2 dx = I_2\,, \tag{57}$$

which can be used to define the velocity of the centre of gravity of $q(x,y)$ because the left side of Eq. (57) is independent of time as I_2 is a conserved quantity and thus time-independent. With the centre of gravity X_E of $q(x,t)$ defined as:

$$x_E = \frac{\int_{-\infty}^{\infty} x|q|^2 dx}{\int_{-\infty}^{\infty}|q|^2 dx}\,, \quad \text{it follows that} \quad \frac{dx_E}{dt} = \frac{I_2}{I_1}\,, \tag{58}$$

and thus x_E is also a conserved quantity.

Ablowitz and Segur (1979) gave the following conserved quantities belonging to the D&S equations (42),

$$I_1 = \iint |B|^2 d\xi d\eta\,, \tag{59a}$$

$$I_2 = \iint \left(B\frac{\partial B^*}{\partial \xi} - A^*\frac{\partial A}{\partial \xi}\right)d\xi d\eta\,, \tag{59b}$$

$$I_3 = \iint \left(B\frac{\partial B^*}{\partial \eta} - A^*\frac{\partial A}{\partial \eta}\right)d\xi d\eta\,, \tag{59c}$$

$$I_4 = \iint \left[\left\{ \lambda_1 \left| \frac{\partial B}{\partial \xi} \right|^2 + \mu_1 \left| \frac{\partial A}{\partial \eta} \right|^2 \right\} \right.$$
$$\left. - \frac{1}{2} \left\{ -\chi_1^\phi |B|^4 + \frac{\alpha \chi_2}{\kappa_2^\phi} \left(\frac{\partial \bar{\phi}}{\partial \xi} \right)^2 + \frac{(\chi_2^\phi)^3}{\kappa_2^\phi} \left(\frac{\partial \bar{\phi}}{\partial \eta} \right)^2 \right\} \right] d\xi d\eta \,. \qquad (59d)$$

These conservation laws can be useful as a check on numerical computations of NLS-type equations.

3.5. *Special cases of NLS-type equations*

In the reduction to the 1D case, the dependence on the lateral coordinate η is ignored. Equation (42b) can then be integrated once so that the resulting system is:

$$i \frac{\partial B}{\partial \tau} + \lambda_1 \frac{\partial^2 B}{\partial \xi^2} = \nu_1 |B|^2 B \,, \qquad (60a)$$

$$\alpha \frac{\partial \phi^{(1,0)}}{\partial \xi} = -\kappa_2 |B|^2 \,, \qquad (60b)$$

where the condition $\partial \phi^{(1,0)} / \partial \xi = 0$ whenever $|B| = 0$ has been used and ν_1 is given by:

$$\nu_1 = \chi_1 - \frac{\kappa_2}{\alpha} \chi_2 \,. \qquad (60c)$$

The long-wave potential $\phi^{(1,0)}$ can be determined after the complex amplitude has been determined from the NLS equation (60a).

In the limit for deep water, we have $kh \to \infty$ and $\sigma \to 1$. Keeping now $\varepsilon kh \ll 1$ (and, therefore, in fact, $\delta kh \ll 1$, or the length of the wave group is much larger than the depth), Eq. (40) reduces to $\nabla \phi^{(1,0)} = 0$ and Eq. (41) reduces to:

$$2i\omega \frac{\partial B}{\partial \tau} - \frac{g}{4k} \frac{\partial^2 B}{\partial \xi^2} + \frac{g}{2k} \frac{\partial^2 B}{\partial \eta^2} = 4k^4 |B|^2 B \,, \qquad (61)$$

which is the two-dimensional nonlinear Schrödinger equation for deep water. When the envelope B is independent of the lateral coordinate η, the equation reduces further to the one-dimensional NLS equation for deep water. Here, the envelope B is the amplitude of the potential. Notice that Eq. (61) is in fact uniformly valid for $kh \to \infty$ (the condition $\varepsilon kh \ll 1$ need not be imposed) because the troublesome terms cancel in the next approximation, see Longuet–Higgins (1976).

Davey and Stewartson (1974) gave the following set of equations as the shallow water limit ($kh \to 0$ and $c_g \to c$) of Eqs. (40) and (41),

$$\frac{2ik}{\sqrt{gh}}\frac{\partial B}{\partial \tau} - (kh)^2\frac{\partial^2 B}{\partial \xi^2} + \frac{\partial^2 B}{\partial \eta^2} = \frac{9k^2}{2gh^3}|B|^2 B + \frac{3k^2}{\sqrt{gh}} B \frac{\partial \phi^{(1,0)}}{\partial \xi}\,, \tag{62a}$$

$$(kh)^2\frac{\partial^2 \phi^{(1,0)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(1,0)}}{\partial \eta^2} = -\frac{3k^2}{\sqrt{gh}}\frac{\partial |B|^2}{\partial \xi}\,. \tag{62b}$$

The validity of these equations has been checked by Freeman and Davey (1975) who started from the original equations of motion and introduced the appropriate long-wave scaling. They showed that the double limit $\varepsilon \to 0$, $kh \to 0$ in the asymptotic expansions for Φ and ζ was uniform when the condition $\varepsilon/(kh)^2 \ll 1$ was fulfilled. Because $\varepsilon = ka$, this condition can be written as $(a/h)/(kh)$. This condition does not necessarily imply that the Stokes parameter which is given by $(a/h)/(kh)^2$ is a small quantity.[a]

Under the condition that $\varepsilon/(kh)^2 = \mathcal{O}(1)$, Freeman and Davey obtained the Kadomtsev–Petviashvillii equation (the two-dimensional generalisation of the Korteweg de Vries equation).

When B and $\phi^{(1,0)}$ are independent of η, i.e. B and $\phi^{(1,0)}$ do not vary in the direction perpendicular to the wave motion, the set of shallow-water equations reduces to a NLS equation for B,

$$\frac{2ik}{\sqrt{gh}}\frac{\partial B}{\partial \tau} - (kh)^2\frac{\partial^2 B}{\partial \xi^2} = -\frac{9k^2}{2gh^3}|B|^2 B\,, \tag{63}$$

here use has been made of $\partial\phi^{(1,0)}/\partial\xi = 0$ for $|B| = 0$.

3.6. *Effects of surface tension*

Djordjevic and Redekopp (1977) extended the analysis of Davey and Stewartson (1974) in that they also included the effect of surface tension. The linear dispersion relation and the phase and group velocities then are given by:

$$\omega^2 = (1+\tilde{\gamma})gk\tanh kh\,, \quad \text{with} \quad \tilde{\gamma} = \frac{\gamma k^2}{\rho g}\,, \tag{64a}$$

[a]Notice that Djordjevic and Redekopp (1978) defined $\varepsilon/(kh)^2$ to be the Stokes parameter while ε measured the wave slope ka.

$$c = \left[\frac{g}{k}((1+\tilde{\gamma})\tanh kh\right]^{\frac{1}{2}}, \tag{64b}$$

$$c_g = \frac{1}{2}c\left[1 + kh\frac{1-\sigma^2}{\sigma} + 2\frac{\tilde{\gamma}}{1+\tilde{\gamma}}\right]. \tag{64c}$$

It is found that the second-harmonic term $\zeta^{(2,2)}$ becomes singular when $\tilde{\gamma} = \sigma^2/(3-\sigma^2)$. In that case, one has second-harmonic resonance which is possible for capillary waves. Assuming that the wave numbers are not too close to the ones for which $\tilde{\gamma} = \sigma^2/(3-\sigma^2)$, the third-order terms may be considered. Instead of Eq. (40), now the following equation is obtained for the leading-order mean flow $\phi^{(1,0)}$,

$$(gh - {c_g}^2)\frac{\partial^2\phi^{(1,0)}}{\partial\xi^2} + gh\frac{\partial^2\phi^{(1,0)}}{\partial\eta^2} = -k^2\left\{c_g(1-\sigma^2) + \frac{2}{1+\tilde{\gamma}}c\right\}\frac{\partial|B|^2}{\partial\xi}. \tag{65}$$

Introducing a quantity Q by[b]:

$$Q = \frac{c_g}{k^2}\frac{\partial\phi^{(1,0)}}{\partial\xi} + \frac{c_g}{gh - {c_g}^2}\left\{2c\frac{\tilde{\gamma}}{1+\tilde{\gamma}} + c_g(1-\sigma^2)\right\}|B|^2, \tag{66}$$

the set of Eqs. (48) is recovered now with the coefficients[c] modified by the surface tension effect, see Djordjevic and Redekopp (1977), Ablowitz and Segur (1979, 1981, p. 320) or Dingemans (1997, p. 903).

A few notable differences between the situation with and without surface tension are listed below,

- With surface tension included, it becomes possible that $c_g > gh$. This is easiest seen by considering a series expansion for small kh of c_g from Eq. (64c) which yields,

$$c_g = \sqrt{gh(1+\tilde{\gamma})}\left(1 + \frac{\tilde{\gamma}}{1+\tilde{\gamma}}\right) = \mathcal{O}((kh)^2). \tag{67}$$

  As $\tilde{\gamma} > 0$, we have $c_g > \sqrt{gh}$ in the shallow-water approximation.

[b]Notice that in Eq. (2.14) of Djordjevic and Redekopp (1977), a printing error occurs: the first term between curly brackets has $2c$ instead of $2c\tilde{\gamma}$ in the numerator.
[c]Notice that a misprint in Djordjevic and Redekopp (1978, Eq. (2.17)) is present: the third term between square brackets in the expression for ν_1 should have the numerical factor one and not four, see also Eqs. (48).

- The coefficient ν_1 of the term $|B|^2 B$ is singular when $c_g = \sqrt{gh}$ and when $\tilde{\gamma} = \sigma^2/(3-\sigma^2)$. For $c_g = \sqrt{gh}$, one has long-wave/short-wave resonance in which the group velocity of the short (capillary) waves equals the phase velocity of the long (gravity) waves. Near ${c_g}^2 = gh$, another scaling of the independent variables has to be used and the asymptotic expansion of Φ and ζ is also different (Djordjevic and Redekopp, 1977),

$$\Phi = \varepsilon^{2/3}\phi_0 + \varepsilon\phi_1 + \varepsilon^{4/3}\phi_2 + \cdots\,, \tag{68a}$$

$$\zeta = \varepsilon^{2/3}\zeta_0 + \varepsilon\zeta_1 + \varepsilon^{4/3}\zeta_2 + \cdots\,, \tag{68b}$$

with

$$\zeta = \varepsilon^{2/3}(x_1 - c_g t)\,, \quad \eta = \varepsilon^{2/3}x_2\,, \quad \tau = \varepsilon^{4/3}t\,. \tag{68c}$$

In this case, the 1D version becomes,

$$i\frac{\partial B}{\partial \tau} + \lambda_1 \frac{\partial^2 B}{\partial \xi^2} = FB\,, \quad \frac{\partial F}{\partial \tau} = -\alpha_1 \frac{\partial}{\partial \xi}(|B|^2)\,, \tag{69}$$

where

$$F(\xi,\tau) = \delta_1 \frac{\partial \phi^{(1,0)}}{\partial \xi}\,,$$

and

$$\delta_1 = k\left[1 + \frac{c_g}{2c}(1-\sigma^2)(1+\tilde{\gamma})\right] \quad \text{and} \quad \alpha_1 = \frac{1}{2}\delta_1 k^2(1-\sigma^2) > 0\,.$$

A major aid in discussing the general behaviour of the equation system including surface tension is the figure for the parameter space of the coefficients with respect to the surface tension parameter $\tilde{\gamma} = \gamma k^2/(\rho g)$ and kh (Fig. 1, taken from Ablowitz and Segur, 1981). A similar picture is also given by Djordjevic and Redekopp (1977). The corresponding set of equations is Eq. (42) where, as noted before, the coefficients are modified due to the effect of surface tension. An explicit definition of these coefficients are found in Ablowitz and Segur (1981, p. 320) with $\nu \equiv \nu_1 = \chi_1 - \kappa_2\chi_2/\alpha$ and $\lambda \equiv \lambda_1$, $\chi \equiv \chi_1$.

The character of the solution is determined by the sign of the coefficient. Therefore, each region of this figure has specific characteristics regarding the dynamical behaviour of waves. Along the two lines bounding the region F, singularities of ν occurs. The other three lines represent a simple zero of a coefficient. The regions C, D and F where $\lambda\nu < 0$ admit solitons as solutions for

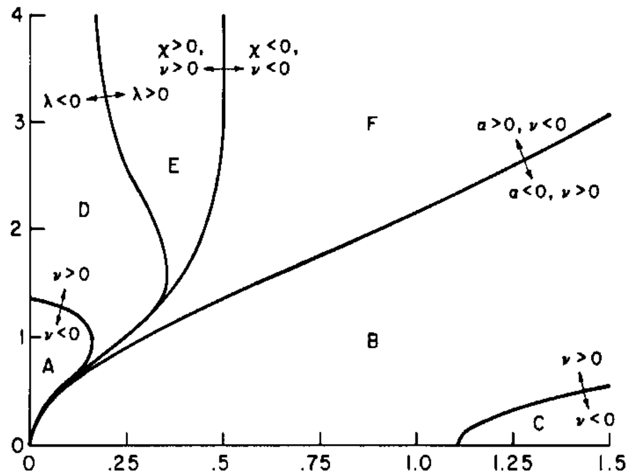

Fig. 1. Parameter space of the coefficients inclusive surface tension, showing where they change sign as functions of surface tension $\tilde{\gamma}$ (abscissa) and kh (ordinate); from Ablowitz and Segur (1981).

the amplitude envelope. No soliton solutions are possible in the regions A, E and B where $\lambda\nu > 0$. The reader would find it worthwhile to refer to Ablowitz and Segur (1979, 1981) and Djordjevic and Redekopp (1977) for an elaborate discussion on the behaviour of the solutions in the separate regions. For water waves for which surface tension effect is negligible, one is primarily interested along the ordinate ($\tilde{\gamma} = 0$). Some more properties of the system along this ordinate are discussed in section 5.

4. Nonlinear Schrödinger-type Equations: Uneven Bottom

4.1. *Propagation in one dimension*

In the same way as a weakly-dispersive long-wave equation such as the KdV equation for water of constant depth can be extended to a KdV-like equation for the case of varying depth, $h = h(x)$, an inhomogeneous NLS equation can be derived for the propagation of wave packets on an uneven bottom. In both cases, the reflection has to be neglected because both the NLS and the KdV equations describe waves propagating in one direction only. Djordjevic and Redekopp (1978) gave a derivation of an inhomogeneous NLS equation in a way which is similar to that in which the Davey and Stewartson equations