



Fig. 1. Parameter space of the coefficients inclusive surface tension, showing where they change sign as functions of surface tension $\tilde{\gamma}$ (abscissa) and kh (ordinate); from Ablowitz and Segur (1981).

the amplitude envelope. No soliton solutions are possible in the regions A, E and B where $\lambda\nu > 0$. The reader would find it worthwhile to refer to Ablowitz and Segur (1979, 1981) and Djordjevic and Redekopp (1977) for an elaborate discussion on the behaviour of the solutions in the separate regions. For water waves for which surface tension effect is negligible, one is primarily interested along the ordinate ($\tilde{\gamma} = 0$). Some more properties of the system along this ordinate are discussed in section 5.

4. Nonlinear Schrödinger-type Equations: Uneven Bottom

4.1. Propagation in one dimension

In the same way as a weakly-dispersive long-wave equation such as the KdV equation for water of constant depth can be extended to a KdV-like equation for the case of varying depth, $h = h(x)$, an inhomogeneous NLS equation can be derived for the propagation of wave packets on an uneven bottom. In both cases, the reflection has to be neglected because both the NLS and the KdV equations describe waves propagating in one direction only. Djordjevic and Redekopp (1978) gave a derivation of an inhomogeneous NLS equation in a way which is similar to that in which the Davey and Stewartson equations

(48) are derived. The depth is slowly varying, $h = h(\beta^2 x)$ and β is supposed to be proportional to ε where ε is the wave slope which conforms with the common assumption in the NLS scaling. Because $\beta = \lambda/\Lambda$ is the modulation parameter and the modulation of the carrier wave gives rise to a wave group, Λ may be seen as a measure for the horizontal extent of the wave group. Because the group velocity is a function of the depth h and therefore also a function of $\varepsilon^2 x$, the following multiple scales are introduced now,

$$\tau = \varepsilon \left\{ \int^x \frac{dx}{c_g(\xi)} - t \right\}, \quad \xi = \varepsilon^2 x. \tag{70}$$

Note that the role of τ and ξ is reversed compared to the constant-depth case. It is supposed that $\omega = \text{constant}$, i.e., no temporal variation of the medium is considered. It is supposed that the group velocity c_g , the phase velocity c and the wave number k can locally be defined as a function of the local depth $h(\xi)$ and therefore the variation of c_g , c and k is with ξ : $c_g(\xi)$, $c(\xi)$ and $k(\xi)$. The free surface is expanded as:

$$\zeta(x, t) = \sum_{n=1}^{\infty} \left\{ \sum_{m=-n}^n \zeta^{(n,m)}(\xi, \tau) E^m \right\}, \tag{71a}$$

with

$$E = \exp \left\{ i \left[\int^x k(\xi) dx - \omega t \right] \right\}, \tag{71b}$$

and

$$\zeta^{(n,-m)} = (\zeta^{(n,m)})^*, \tag{71c}$$

with $()^*$ denoting the complex conjugate. This expansion thus is similar with the expansion used before, the difference being an adoption to the nonuniform depth which necessitates the adoption of a coordinate moving with a nonuniform velocity so as to remain near the centre of the wave group. Proceeding in the usual way, we find from the $\varepsilon^3 E^0$ terms an equation for $\phi^{(1,0)}$ and from the $\varepsilon^3 E^1$ terms an equation for the amplitude $B(\xi, \tau)$ of the solution for $\phi^{(1,1)}$ emerges. Introducing the quantity $Q(\xi, \tau)$ by:

$$gQ(\xi, \tau) = \frac{\partial \phi^{(1,0)}}{\partial \tau} + \frac{k^2 c_g}{gh - c_g^2} \left\{ 2 \frac{c_g}{c} + 1 - \sigma^2 \right\} |B|^2, \tag{72}$$

the equation for $\phi^{(1,0)}$ becomes simply,

$$\frac{\partial Q}{\partial \tau} = 0, \quad \text{with the solution } Q = Q_0(\xi), \tag{73}$$

and the equation for $B(\xi, \tau)$ becomes,

$$i \frac{\partial B}{\partial \xi} + \lambda_1 \frac{\partial^2 B}{\partial \tau^2} + i\mu_1 B = \nu_1 |B|^2 B + \nu_2 Q_0 B, \tag{74a}$$

where the coefficients are given by^d:

$$\mu_1 = \frac{(1 - \sigma^2)(1 - kh\sigma)}{\sigma + kh(1 - \sigma^2)} \cdot \frac{d(kh)}{d\xi} = -\frac{\sigma(1 - kh\sigma)}{\sigma + kh(1 - \sigma^2)} \cdot \frac{k'}{k}, \tag{74b}$$

$$\lambda_1 = -\frac{1}{2\omega c_g} \left\{ 1 - \frac{gh}{c_g^2} (1 - kh\sigma)(1 - \sigma^2) \right\} \equiv \frac{1}{2c_g^3} \frac{\partial^2 \omega}{\partial k^2}, \tag{74c}$$

$$\nu_1 = \frac{k^4}{4\omega\sigma^2 c_g} \left[9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2 c_g^2}{hg - c_g^2} \cdot \left\{ 4 \left(\frac{c}{c_g} \right)^2 + 4 \frac{c}{c_g} (1 - \sigma^2) + \frac{gh}{c_g^2} (1 - \sigma^2)^2 \right\} \right], \tag{74d}$$

$$\nu_2 = \frac{k^2}{2\sigma} \cdot \frac{c}{c_g} \left[2 \frac{c}{c_g} + 1 - \sigma^2 \right] \geq 0. \tag{74e}$$

It is noted that Eq. (74a) can be written in a simpler form upon application of the transformation,

$$B(\xi, \tau) = \tilde{B}(\xi, \tau) \exp \left[-i \int^\xi \nu_2(\tilde{\xi}) Q_0(\tilde{\xi}) d\tilde{\xi} \right]. \tag{75}$$

The resulting inhomogeneous NLS equation then reads in terms of \tilde{B} ,

$$i \frac{\partial \tilde{B}}{\partial \xi} + \lambda_1 \frac{\partial^2 \tilde{B}}{\partial \tau^2} - \nu_1 |\tilde{B}|^2 \tilde{B} = -i\mu_1 \tilde{B}. \tag{76}$$

It is obvious that the term $\nu_2 Q_0 B$ gives only a phase shift. The essential difference between the NLS equation (26) and (76) is the term $-i\mu_1 \tilde{B}$ in the right-hand side of Eq. (76). Another difference is that the coefficients λ_1 and ν_1 in Eq. (76) are functions of ξ .

Equation (76) describes the evolution of wave packets propagating over an uneven bottom under the condition that reflection can be neglected and

^dNote that $(1 - \sigma^2)^2$ occurs in the last term between curly brackets in the expression for ν_1 and not $(1 - \sigma)^2$ as given by Djordjevic and Redekopp (1978, Eq. (2.17)).

consequently the depth varies very slowly, $h = h(\beta^2 x)$. Only for constant depth, we have $\mu_1 = 0$ and λ_1, ν_1 are constants.

Equation (76) can be transformed to a homogeneous equation with ξ -dependent coefficients by introducing the transformation,

$$\tilde{B}(\xi, \tau) = \alpha(\xi)D(\xi, \tau). \tag{77}$$

By carrying out the transformation and choosing $\alpha^{-1}d\alpha/d\xi = -\mu_1$, one obtains,

$$i \frac{\partial D}{\partial \xi} + \lambda_1 \frac{\partial^2 D}{\partial \tau^2} - \tilde{\nu}_1 |D|^2 D = 0, \tag{78a}$$

with

$$\tilde{B} = \alpha D, \quad \tilde{\nu}_1 = \alpha \nu_1 \quad \text{and} \quad \alpha(\xi) = \exp \left[- \int^\xi \mu_1(\tilde{\xi}) d\tilde{\xi} \right]. \tag{78b}$$

4.2. Propagation in two horizontal dimensions

The starting point for the derivation is the usual set of equations for inviscid, irrotational fluid motion with a free surface $z = \zeta(\mathbf{x}, t)$ and velocity potential $\Phi(\mathbf{x}, z, t)$ on water of varying depth with the bottom given by $z = -h(\mathbf{x})$.

For the derivation of the evolution equations for complex amplitude of the wave group and the accompanying long wave motion is referred to Liu and Dingemans (1989) and the references referred to there. Basic to the perturbation approach is the introduction of two small parameters viz. a modulation parameter δ and a nonlinearity parameter ε . The nonlinearity parameter is related to the slope of the carrier wave ka with a and k the amplitude and wave number of the carrier wave. The modulation parameter is related to both the inhomogeneity of the medium, i.e., the bottom slope and the variation of the incoming wave field in both time and space. Use is made of slow scales $\mathbf{x}_1 = \delta \mathbf{x} = [\delta x, \delta y]$ and $t_1 = \delta t$ so that $\nabla_1 = \partial/\partial \mathbf{x}_1$ and the most general set of equations is obtained when ε and δ are of equal order.

The free surface elevation $\zeta(\mathbf{x}, t)$ and velocity potential $\Phi(\mathbf{x}, z, t)$ are expanded in terms of the nonlinearity parameter ε and, as we are interested in the propagation of harmonic waves, also in terms of harmonics,

$$\zeta = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{m=+n} \tilde{\zeta}^{(n,m)} E^m, \quad \Phi = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{m=+n} \tilde{\phi}^{(n,m)} E^m, \tag{79}$$

with $E = \exp \left[\frac{i}{\varepsilon} \chi(\mathbf{x}_1, t_1) \right],$

and $\tilde{\zeta}^{(n,-m)}$ is the complex conjugates of $\tilde{\zeta}^{(n,m)}$ and similarly for $\tilde{\phi}^{(n,-m)}$. Expansion of the phase function χ as $\chi = \sum_{n=0} \varepsilon^n \chi_n(\mathbf{x}_1, t_1)$ and introducing new functions $\zeta^{(n,m)}$ by:

$$\zeta^{(n,m)} = \tilde{\zeta}^{(n,m)} \exp \left[im \sum_{n=1} \varepsilon^n \chi_n \right] \quad \text{and similarly for } \phi^{(n,m)},$$

gives the expansions,

$$\zeta = \sum_{n=1} \varepsilon^n \sum_{m=-n}^{m=+n} \zeta^{(n,m)} E_0^m, \quad \Phi = \sum_{n=1} \varepsilon^n \sum_{m=-n}^{m=+n} \phi^{(n,m)} E_0^m, \tag{80}$$

where $E_0 = \exp[i\chi_0(\mathbf{x}_1, t_1)/\varepsilon]$,

and $\omega_0 = -\partial\chi_0/\partial t_1$ is the *constant* carrier wave frequency and $\mathbf{k}_0 = \nabla_1\chi_0$ is related to ω_0 through the linear dispersion relationship $\omega_0^2 = gk_0 \tanh k_0 h$ with $k_0 = |\mathbf{k}_0|$. For the several orders in $\varepsilon(n)$ and the harmonics (m) equations for the $\zeta^{(n,m)}$ and $\phi^{(n,m)}$ are obtained. For the first-order problem is obtained,

$$\frac{\partial\phi^{(1,0)}}{\partial z} = \zeta^{(1,0)} = 0, \quad \phi^{(1,1)} = -i \frac{gA \cosh[k_0(h+z)]}{2\omega_0 \cosh(k_0 h)}, \quad \zeta^{(1,1)} = \frac{1}{2}A, \tag{81}$$

where A is an unknown complex amplitude.

To ensure the nonsecularity of the higher-order solutions, solvability conditions have to be imposed, see Chu and Mei (1970) and Liu and Dingemans (1989). The final result is an evolution equation for the complex amplitude \bar{A} and a wave equation with forcing for the wave induced current $\phi^{(1,0)}$ which is a real function (see Eqs. (5.12) and (5.14) of Liu and Dingemans, 1989). These equations are, without the fast varying part h_1 of the bottom, and in physical variables, while writing ϕ for $\phi^{(1,0)}$ and A for \bar{A} for convenience,

$$\begin{aligned} & \frac{\partial A}{\partial t} + \mathbf{c}_g \cdot \nabla A + \frac{1}{2}(\nabla \cdot \mathbf{c}_g)A - \frac{i}{2} \left\{ \frac{\mathbf{k}_0}{k_0} \nabla \left(\frac{\partial c_g}{\partial k_0} \frac{\mathbf{k}_0}{k_0} \cdot \nabla \right) A \right. \\ & \quad \left. + \nabla \cdot \left(\frac{c_g}{k_0} \nabla A \right) - \frac{\mathbf{k}_0}{k_0} \cdot \nabla \left(\frac{c_g}{k_0} \frac{\mathbf{k}_0}{k_0} \cdot \nabla \right) A - \mu \frac{\mathbf{k}_0}{k_0} \cdot \nabla A \right\} \\ & \quad - i \left(\frac{\omega_0 k_\infty}{2g} (\sigma^2 - 1) \frac{\partial \phi}{\partial t} - \mathbf{k}_0 \cdot \nabla \phi \right) A \\ & \quad + ik_0^2 \omega_0 \kappa |A|^2 A + \frac{i}{2} \nu A = 0, \end{aligned} \tag{82a}$$

and

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (gh \nabla \phi) = \nabla \cdot \left[\frac{\mathbf{k}_0}{2\omega_0} g^2 |A|^2 \right] - \frac{\omega_0^2}{4} \frac{\partial}{\partial t} \left(\frac{|A|^2}{\sinh^2 q} \right), \tag{82b}$$

where

$$\kappa = \frac{1}{16 \sinh^4 q} (\cosh 4q + 8 - 2 \tanh^2 q), \quad k_\infty = \frac{\omega_0^2}{g}, \tag{83a}$$

$$\sigma = \coth q \quad \text{and} \quad q = k_0 h, \tag{83b}$$

and the coefficients μ and ν represent functions of the derivatives of depth and wave number of which the expressions have been given in Liu and Dingemans (1989, Eqs. (B.1) and (B.2)). Both μ and ν are zero in case of a horizontal bottom. Notice that $\nabla \equiv (\partial_x, \partial_y)^T$ is the horizontal gradient operator.

For a horizontal bottom, the evolution equation, Eq. (82a), simplifies considerably. Taking the main wave direction in the x -direction (and thus \mathbf{k}_0 is directed along the x -axis so that $\mathbf{k}_0 \cdot \nabla = k_0 \partial/\partial x$), the resulting evolution equation reads for horizontal bottom,

$$\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} - \frac{i}{2} \left\{ \frac{\partial c_g}{\partial k_0} \frac{\partial^2 A}{\partial x^2} + \frac{c_g}{k_0} \frac{\partial^2 A}{\partial y^2} \right\} + ik_0^2 \omega_0 \kappa |A|^2 A - iAG\phi = 0, \tag{84a}$$

with

$$G\phi = \left(\frac{\omega_0 k_\infty}{2g} (\sigma^2 - 1) \frac{\partial}{\partial t} - k_0 \frac{\partial}{\partial x} \right) \phi. \tag{84b}$$

The corresponding wave equation for a horizontal bottom with \mathbf{k}_0 directed along the x -axis is:

$$\frac{\partial^2 \phi}{\partial t^2} - gh \nabla^2 \phi = \frac{k_0 g^2}{2\omega_0} \frac{\partial |A|^2}{\partial x} - \frac{\omega_0^2}{4 \sinh^2 q} \frac{\partial |A|^2}{\partial t}. \tag{85}$$

The reduction of Eq. (82) to 1D is readily obtained by supposing \mathbf{k}_0 is directed along tree x -axis and ignoring all y -dependence,

$$\begin{aligned} i \left(\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} \right) + \frac{1}{2} \frac{\partial c_g}{\partial k_0} \frac{\partial^2 A}{\partial x^2} + AG\phi - k_0^2 \omega_0 \kappa |A|^2 A = \\ - \frac{1}{2} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial c_g}{\partial k_0} \right) - \mu \right\} \frac{\partial A}{\partial x} - \left(\frac{i}{2} \frac{\partial c_g}{\partial x} - \frac{1}{2} \nu \right) A, \end{aligned} \tag{86a}$$

and

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial x} \left(gh \frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{k_0}{2\omega_0} g^2 |A|^2 \right) - \frac{\omega^2}{4} \frac{\partial}{\partial t} \left(\frac{|A|^2}{\sinh q} \right). \tag{86b}$$

4.3. Shallow-water limit

In the shallow-water limit, $kh \rightarrow 0$, we obtain, under the condition that the Stokes number is small, $\varepsilon/(kh)^2 \ll 1$, the following expressions for the coefficients μ_1 , λ_1 , ν_1 and ν_2 ,

$$\mu_1 = -\frac{1}{2} \frac{k'}{k} = \frac{1}{4} \frac{h'}{h}, \quad \lambda_1 = -\frac{\omega}{2g^{3/2}} h^{1/2}, \tag{87a}$$

$$\nu_1 = -\frac{9}{4} \frac{\omega}{g^{3/2}} h^{-7/2}, \quad \nu_2 = \frac{3}{2} \frac{\omega}{g^{1/2}} h^{-3/2}, \tag{87b}$$

where a prime denotes differentiation to the argument.

For $\lambda_1 \nu_1 > 0$ (which is always the case in shallow water), the governing equations for the envelope-hole solution reduce to the inhomogeneous KdV equation with variable coefficients. That Eq. (78a) reduces to this generalised KdV equation can be shown in the following way. Write $B = R(\xi, \tau) \exp[i \int^\tau \theta(\xi, \tilde{\tau} d\tilde{\tau})]$ with R and θ be the real functions. R and θ are then expanded in a power series to a small parameter δ : $R = R_0 + \delta R_1 + \delta^2 R_2 + \dots$ and $\theta = \delta \theta_1 + \delta^2 \theta_2 + \dots$ where δ is a measure of the slope of the modulation of a wave train about a uniform finite-amplitude state. The discussion here is restricted to any small-amplitude long-wave perturbation of a wave which is modulationally stable (i.e., $\lambda_1 \nu_1 > 0$). The following further coordinate stretching is introduced,

$$T = \delta^{1/2} \left\{ \int^\xi \frac{d\tilde{\xi}}{c(\tilde{X})} - \tau \right\}, \quad X = \delta^{3/2} \xi. \tag{88}$$

Substitution of the expansions for R and θ yields expressions for $R_0(X)$ and $c(X)$ and a relation between $\theta_1(X, T)$ and $R_1(X, T)$. The secularity condition for R_2 and θ_2 yields the generalised KdV equation which can be written with $R_1 = h^{7/2} H(X, T)$ as:

$$\frac{\partial H}{\partial X} - \frac{9\omega_0}{2g^{3/2}} h^{-5/8} H \frac{\partial H}{\partial T} + \frac{\omega_0}{12r_0 g^{3/2}} h^{11/4} \frac{\partial^3 H}{\partial T^3} = 0, \tag{89}$$

where $R_0 = r_0 h^{-1/4}$.

4.4. Effect of an ambient current on 1D propagation

We proceed with the assumption of $kh = \mathcal{O}(1)$ and a current U such that $U/\sqrt{gh} \leq 1$. In the absence of waves, the current variation may be determined

by the nonlinear shallow water equation, i.e.,

$$\frac{\partial \zeta^c}{\partial t} + \frac{\partial U(h + \zeta^c)}{\partial x} = 0, \tag{90}$$

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial \zeta^c}{\partial x} = 0. \tag{91}$$

Further, the variation of depth and current is assumed to be an order of magnitude higher than the wave nonlinearity parameter $\varepsilon = ka$, i.e.,

$$\frac{1}{kh} \frac{dh}{dx} = \frac{1}{kU} \frac{\partial U}{\partial x} = \frac{1}{\omega U} \frac{\partial U}{\partial t} = \mathcal{O}(\varepsilon^2). \tag{92}$$

Using a perturbation analysis with respect to the underlying current (U, ζ^c), Turpin *et al.* (1983) showed that the amplitude A of the waves satisfies the equation (note that the form of the equation mentioned below is a result of multiplying ic_g to their original Eq. (2.22) and converting the pressure amplitude to surface amplitude),

$$iA_T + ic_g A_\xi + \lambda_1 A_{\tau\tau} - \nu_1 |A|^2 A + i\gamma_c A = 0, \tag{93}$$

where $T = \varepsilon^2 t$, $\xi = \varepsilon^2 x$ and $\tau = \varepsilon(\int(dx/c_g) - t)$. In Eq. (93), A is the complex amplitude of the first order elevation, identical to $\zeta^{(1,1)}$ in the expansion series of Eq. (71). The coefficients vary slowly as a function of depth and current and are given by:

$$\begin{aligned} \lambda_1(\xi, T) &= -\frac{1}{2\omega_r} \frac{(c_g)^2}{(c_g + U)^2} \left[1 - \frac{gd}{c_g^2} (1 - \sigma^2) \cdot (1 - \sigma kd) \right] \\ &= \frac{1}{2(c_g + U)^2} \frac{\partial^2 \omega_r}{\partial k^2}, \end{aligned} \tag{94a}$$

$$\begin{aligned} \nu_1(\xi, T) &= \frac{g^2 k^4}{4\omega_r^3 \sigma^2} \left\{ 9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2 c_g^2}{gd - c_g^2} \right. \\ &\quad \left. \times \left[4 \left(\frac{c}{c_g} \right)^2 + 4 \left(\frac{c}{c_g} \right) (1 - \sigma^2) + \frac{gd}{c_g^2} (1 - \sigma^2)^2 \right] \right\}, \end{aligned} \tag{94b}$$

$$\gamma_c(\xi, T) = -\frac{1}{2\omega_r} \frac{\partial \omega_r}{\partial T} + \frac{\omega_r}{2} \frac{\partial}{\partial \xi} \left(\frac{c_g + U}{\omega_r} \right), \tag{94c}$$

with $\sigma = \tanh(kd)$ and $d = h + \zeta^c$.

The effect of current in the first instance is reflected in the linear dispersion relation, i.e.,

$$\omega = \omega_r + kU \quad \text{with} \quad \omega_r^2 = gk \tanh kd, \quad (95a)$$

$$c = \frac{\omega_r}{k}, \quad v_g = c_g + U, \quad \text{and} \quad (95b)$$

$$c_g = \frac{\partial \omega_r}{\partial k} = \frac{g}{2\omega_r} [\sigma + kh(1 - \sigma^2)], \quad (95c)$$

where v_g is the absolute group velocity, taken with respect to a fixed reference frame and d represents the mean water level including the set-down due to current and ω_r the apparent frequency for an observer moving with the current U .

Equation (93) with the coefficients defined by Eq. (94)^e is an extension of the equation derived by Djordjevic and Redekopp (1978) and represents a general one-dimensional modulation equation for narrow-banded short waves on an ambient current or a long wave (long enough compared to the short waves to validate the scales of Eq. (92)). For an opposing current, waves are prevented from propagating upstream as the group velocity c_g becomes zero. The equation fails near such points and its validity is limited to milder opposing current such that the blocking condition is not met.

Additional properties of the modulation equation, Eq. (93), can be derived from a simplified form made possible through a transformation (Djordjevic and Redekopp, 1978; Turpin *et al.*, 1983). An important parameter that emerges is:

$$K = -\frac{\nu_1}{\lambda_1} s^2; \quad s = \left[\left(\frac{c_g}{\omega_r} \right)_1 \middle/ \left(\frac{c_g}{\omega_r} \right) \right]^{\frac{1}{2}}, \quad (96)$$

where s may be recognised as the shoaling factor for infinitesimal waves. For a given variation of depth and current, broad features of the evolution of a wavepacket may be determined from the parameter K . Formation of a soliton is expected with increase of K to a positive value. In both cases of with and without current, K becomes zero when $k(h + \zeta^c) = 1.36$ where k is the local wave number taking into account the ambient current, if present.

^eA in Eq. (93) is the complex amplitude of surface elevation unlike in the expression ((2.23c), p. 5) in Turpin *et al.* (1983) where A corresponds to the pressure amplitude.